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A novel algorithm for calculating the vertical velocity of incompressible flow based on variation

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ABSTRACT

In this paper, a novel algorithm for calculating the vertical velocity of incompressible flow based on the variational method is proposed. The partial derivatives of the discrete observation data are calculated by the one-dimensional numerical differential method which is based on the Tikhonov regularization. The new method can overcome the shortcoming that the errors of the vertical velocity calculated by the finite difference method increase with the increase of the grid resolution when the horizontal flow field exists observation errors. Numerical results show that the relative error of the novel method is more than 90% lower than that of the finite difference method when the grid resolution is relatively high. Also, when the magnitude of the observation error is unknown or the boundary condition of the vertical velocity is missing, the new method is still superior to the finite difference method.

Keywords: Incompressible flow; Vertical velocity; Variational method; Observation error; Tikhonov regularization

1. Introduction

In the ocean, meteorological observation and some fluid experiments, only the horizontal velocity can be observed but the vertical velocity needs to be solved numerically [1–5]. For incompressible fluids, the continuity Eq. (1) is:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{1}$$

Theoretically, if the horizontal velocities u, v and the vertical velocity boundary conditions are known, the vertical velocity can be obtained by solving the continuity equation numerically. However, when the finite difference method is used, the observation error in the horizontal flow field may lead to a large error in the final vertical velocity. To reduce the error of the calculated vertical velocity, we propose a new method for solving vertical velocity based on the variation.

No matter which numerical method is adopted, it is inevitable to calculate the partial derivatives of the horizontal velocities. In the field of applied mathematics, calculating the derivatives at discrete points is called the numerical differential problem, which is known to be ill-posed in the sense that a small perturbation in the observation data may lead to a large error in the computed derivative, and the error increases sharply as the observation density increases [6,7]. Therefore, if there is an observation error in the horizontal velocity field, a large error will occur when the vertical velocity is calculated by the finite difference method.

For the problem of numerical differentiation, many methods have been proposed [8–11]. The Tikhonov regularization method is one of them, which has been proved to be very effective in solving ill-posed problems and inverse problems [12–22]. Based on the idea of Tikhonov

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regularization, reconstructed the first-order partial derivatives of the two-dimensional meteorological observation data, and calculated the vorticity of the observed wind field using the one-dimensional numerical differential method [23–25]. Their results show that the numerical differential method based on the Tikhonov regularization is feasible to analyze meteorological observation data and can enhance the recognizing ability for the small-scale weather systems.

In this paper, we propose a novel algorithm for calculating the vertical velocity of the incompressible flow based on the variational method. The one-dimensional numerical differential method based on the Tikhonov regularization is used to calculate the partial derivatives [25]. The sensitivity of the new method to the magnitude of the observation error and the boundary conditions of the vertical velocity is studied systematically, and the results of the method are compared with those of the finite difference method.

2. Mathematical theory

2.1. Algorithm for calculating vertical velocity based on variation

In the Cartesian coordinate system, the horizontal observed flow field is $\tilde{u}(x,y,z)$ and $\tilde{v}(x,y,z)$, and the vertical observed component is unknown. The three-dimensional study area is denoted as Ω and the vertical coordinate $z \in [0,H]$. It is assumed that the corrected flow field is (u,v,w), which satisfies the functional

$$J = \iiint_{\Omega} \left[\left(u-\tilde{u}\right)^2 + \left(v-\tilde{v}\right)^2 \right] d\Omega = \min! \tag{2}$$

and the following constraint

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3}$$

where the vertical velocity w satisfies the following boundary condition.

$$w\big|_{z=0} = w_0,\ w\big|_{z=H} = w_H \tag{4}$$

To get the vertical velocity w, the Lagrange multiplier method is used. The generalized functional is defined as

$$\tilde{J} = \iiint_{\Omega} \left\{ \frac{1}{2}\left[\left(u-\tilde{u}\right)^2 + \left(v-\tilde{v}\right)^2 \right] - \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right\} d\Omega = \min! \tag{5}$$

and its variations is

$$\delta\tilde{J} = \iiint_{\Omega} \left[\left(u-\tilde{u}\right)\delta u + \left(v-\tilde{v}\right)\delta v - \lambda\left(\frac{\partial \delta u}{\partial x} + \frac{\partial \delta v}{\partial y} + \frac{\partial \delta w}{\partial z} \right) \right] d\Omega = 0 \tag{6}$$

Applying the Green formula to the above Eq. (7), then

$$\iiint_{\Omega} \left[\left(u-\tilde{u}+\frac{\partial\lambda}{\partial x} \right)\delta u + \left(v-\tilde{v}+\frac{\partial\lambda}{\partial y} \right)\delta v + \frac{\partial\lambda}{\partial z}\delta w \right] d\Omega - \iint_{\partial\Omega} \left(\lambda\delta u, \lambda\delta v, \lambda\delta w\right)\cdot\vec{n}\,dS = 0 \tag{7}$$

where $\vec{n}$ is the outer normal direction of Ω. Using the arbitrariness of δu, δv and δw, it is easy to prove that u, v and λ satisfy the following condition:

$$\begin{cases} u-\tilde{u}+\dfrac{\partial\lambda}{\partial x} = 0 \\ v-\tilde{v}+\dfrac{\partial\lambda}{\partial y} = 0 \\ \dfrac{\partial\lambda}{\partial z} = 0 \\ \lambda\big|_{\partial\Omega} = 0 \end{cases} \tag{8}$$

From Eq. (8) λ is independent of z, namely, $\lambda = \lambda(x,y)$. Substituting Eq. (8) into Eq. (3)

$$\tilde{D} - \Delta\lambda + \frac{\partial w}{\partial z} = 0 \tag{9}$$

where

$$\begin{cases} \tilde{D} = \dfrac{\partial\tilde{u}}{\partial x} + \dfrac{\partial\tilde{v}}{\partial y} \\ \Delta\lambda = \dfrac{\partial^2\lambda}{\partial x^2} + \dfrac{\partial^2\lambda}{\partial y^2} \end{cases} \tag{10}$$

Considering λ is independent of z, and using the vertical velocity boundary conditions (4). Eq. (9) is integrated into the vertical direction and we can get

$$\begin{cases} \Delta\lambda = \dfrac{1}{H}\displaystyle\int_0^H \tilde{D}dz + \dfrac{1}{H}\left(w_H - w_0\right) \\ \lambda\big|_{\partial\Omega} = 0 \end{cases} \tag{11}$$

Eq. (11) is the Dirichlet boundary value problem of Poisson's equation which can be solved by the successive overrelaxation method. From Eq. (8) the corrected horizontal flow field is obtained

$$\begin{cases} u = \tilde{u} - \dfrac{\partial\lambda}{\partial x} \\ v = \tilde{v} - \dfrac{\partial\lambda}{\partial y} \end{cases} \tag{12}$$

Then, the vertical velocity can be obtained by integrating Eq. (9) in the range $[0,z]$ of the vertical direction, that is,

$$w\left(x,y,z\right) = w_0 - \int_0^z \tilde{D}dz + z\Delta\lambda \tag{13}$$

2.2. One-dimensional numerical differential method

In solving Eqs. (11) and (13) it is necessary to calculate the partial derivatives of the observed horizontal velocities, which is a typical numerical differential problem. In this paper, the one-dimensional numerical differential method based on Tikhonov regularization introduced in [25] is used to calculate the first-order partial derivatives. The basic idea of the method is briefly described below.

Let $y = y(x)$ be a function with $x \in [0,1]$, $h_i = x_i - x_{i-1}$ and $h = \max_{1\le i\le n} h_i$. $\tilde{y}_i$ is the observation data at x_i and δ is a given constant which satisfies

$$\left|\tilde{y}_i - y\left(x_i\right)\right| \le \delta,\ i = 0,1,2,\ldots,n \tag{14}$$

We want to find a function $f_*(x)$ that satisfies $f_*'(x)$ is an approximation of $y'(x)$.

Suppose $\tilde{y}_0 = y(0)$ and $\tilde{y}_n = y(1)$. Namely, the values at both ends are considered to be accurate. The following function is defined:

$$J\left(f\right) = \sum_{j=1}^{n-1} \frac{h_j + h_{j+1}}{2}\left(\tilde{y}_j - f\left(x_j\right)\right)^2 + \alpha\left\|f''\right\|^2_{L^2(0,1)} \tag{15}$$

where α the regularization parameter. The above problem can be transformed into the following two problems.

Problem-1: Find the function $f_* \in H^2(0,1)$ satisfying $f_*(0) = y(0)$ and $f_*(1) = y(1)$ such that $J(f_*) = \min!$.

Problem-2: If f_* exists, how to choose the regularization parameter α related to δ, so that $f_*'(x)$ is an approximation $\alpha = \delta^2$ of $y'(x)$?

Theoretically, for any $\alpha > 0$, the solution of Problem-1 exists and is unique, and f_* is a piecewise natural cubic spline function. The solution of Problem-1 is convergent when the regularization parameter is simply taken as. A detailed proof process can refer to [26–27].

The expression of f_* on subinterval $[x_i, x_{i+1}]$ is:

$$f_*\left(x\right) = a_j + b_j\left(x - x_j\right) + c_j\left(x - x_j\right)^2 + d_j\left(x - x_j\right)^3,\ j = 0,1,2,\ldots,n-1 \tag{16}$$

Thus a total of $4n$ undetermined coefficients can be obtained by solving the following Eqs. (17), (18), (19) and (20) and the solution is unique.

$$f_*^{(i)}\left(x_j+\right) - f_*^{(i)}\left(x_j-\right) = 0,\ i = 0,1,2;\ j = 0,1,2,\ldots,n-1 \tag{17}$$

$$f_*^{(3)}\left(x_j+\right) - f_*^{(3)}\left(x_j-\right) = \frac{1}{\delta^2\left(n-1\right)}\left(\tilde{y}_j - f_*\left(x_j\right)\right),\ j = 0,1,2,\ldots,n-1 \tag{18}$$

$$f_*^{(2)}\left(0\right) = 0,\ f_*^{(2)}\left(1\right) = 0 \tag{19}$$

$$f_*\left(0\right) = y\left(0\right),\ f_*\left(1\right) = 0 \tag{20}$$

where $f_*^{(i)}$ denotes the i-th derivative of the function f_*, and

$$f_*^{(i)}\left(x_j+\right) = \lim_{x\to x_j+} f_*^{(i)}\left(x\right),\ f_*^{(i)}\left(x_j-\right) = \lim_{x\to x_j-} f_*^{(i)}\left(x\right) \tag{21}$$

and $\tilde{y}_j$ is the observation data at the j-th point. The specific solution process can refer to [21]. Once these $4n$ coefficients are obtained, the analytical expression of the first-order derivative can be expressed as

$$f_*'\left(x\right) = b_j + 2c_j\left(x - x_j\right) + 3d_j\left(x - x_j\right)^2,\ x \in \left[x_j, x_{j+1}\right],\quad j = 0,1,2,\ldots,n-1 \tag{22}$$

Since the analytical expression, Eq. (16) of the reconstruction function in each subinterval can be obtained, it is noteworthy that the observation field can be reconstructed and refined to any resolution. For the two-dimensional observation data, the first-order partial derivatives can be obtained by the above one-dimensional numerical differentiation method row by row or column by column.

3. Numerical results

3.1. Problem description and calculation scheme

The effectiveness of the method is verified by the following numerical experiments. The three-dimensional incompressible exact flow field in the Cartesian coordinate system is:

$$\begin{cases} u_e = 5\sin\left(x\right)\cos\left(y\right)\cos\left(10z\right) \\ v_e = 5\cos\left(x\right)\sin\left(y\right)\cos\left(10z\right) \\ w_e = -\cos\left(x\right)\cos\left(y\right)\sin\left(10z\right) \end{cases} \tag{23}$$

The observed horizontal flow field $(\tilde{u},\tilde{v})$ is generated by the exact flow field (u_e,v_e) and the uniformly distributed random error in the range of $[-\delta,\delta]$. The following four schemes are proposed to calculate the vertical velocity, and the calculated results are compared with the exact vertical velocity w_e.

Scheme-1: Firstly, the horizontal divergence $\tilde{D}$ of the observation flow field is calculated by the finite difference method. Then, λ it is obtained by solving Eq. (11) and (u,v) is obtained by substituting λ into Eq. (12). Finally, the vertical velocity w is calculated from Eq. (13).

Scheme-2: Firstly, the horizontal divergence $\tilde{D}$ is calculated by the regularization method and the observation flow field $(\tilde{u},\tilde{v})$ is replaced by the reconstructed flow field calculated by Eq. (16). Then, λ it is obtained by solving Eq. (11) and (u,v) is obtained by substituting λ into Eq. (12). Finally, the vertical velocity w is calculated from Eq. (13).

Scheme-3: Firstly, the horizontal divergence $\tilde{D}$ is calculated by the finite difference method. Then, using the finite difference method, the vertical velocity w is obtained by solving the equation $\frac{\partial w}{\partial z} = -\tilde{D}$.

Scheme-4: Firstly, the horizontal divergence $\tilde{D}$ is calculated by the regularization method and the observation flow field is replaced by the reconstructed flow field. Then, using the finite difference method, the vertical velocity w is obtained by solving the equation $\frac{\partial w}{\partial z} = -\tilde{D}$.

The relative error between the calculated vertical velocity and the exact value is defined as:

$$\rho_w = \frac{\left(\sum_{i,j,k}\left(w - w_e\right)^2\right)^{1/2}}{\left(\sum_{i,j,k}\left(w_e^2\right)\right)^{1/2}} \tag{24}$$

where w is the calculated vertical velocity and w_e is the exact value.

3.2. Schemes comparison with determined δ

Let $\delta = 0.05$, $x \in [0,2\pi]$, $y \in [0,2\pi]$ and $z \in [0,0.2\pi]$. Table 1 shows the relative errors calculated at different grid resolutions. As shown in the table, the relative error of Scheme-1 is far less than that of Scheme-3, and the relative error of Scheme-2 is far less than that of Scheme-4. It is shown that the variational-based vertical velocity calculation method is superior to the traditional finite difference method when the calculation method of a derivative is the same.

The relative error of Scheme-1 and Scheme-3 increases with the increase of the resolution, while the relative error of Scheme-2 and Scheme-4 hardly changes with the resolution, which further proves that the problem of numerical differentiation is to be ill-posed in the sense that a small perturbation in the values of the observation flow field may lead to large errors in the computed derivative.

It also can be seen from Table 1 that the relative error of Scheme-2 is less than that of Scheme-1. When the grid resolution is 160 × 160 × 40, the relative error of Scheme-2 is 84% smaller than that of Scheme-1. It is shown that using the regularization method to calculate the derivative can effectively reduce the relative error when the variational-based vertical velocity calculation method is adopted.

In summary, Scheme-2 is the optimal scheme to calculate the vertical velocity. When the grid resolution is 160 × 160 × 40, the relative error of Scheme-2 is reduced by 91% compared with that of the traditional finite difference method, namely Scheme-3.

3.3. Schemes comparison with unknown δ

The regularization parameter in Eq. (15) is taken as $\alpha = \delta^2$. However, the magnitude of the observation error δ is unknown in many practical cases. Therefore, it is necessary to examine the calculation results of Scheme-2 with the change of the guessed observation error δ′. The true value of the observation error is still taken as $\delta = 0.05$, and the regularization parameter is taken as $\alpha = \delta'^2$.

Table 2 shows the relative errors of the vertical velocity obtained by Scheme-2 when δ′ takes different value. It can be seen from the table that even if the exact value δ is unknown, as long as the guess value δ′ is within a certain range, the vertical velocity obtained by the new method is still more accurate than that obtained by the finite difference method. In some cases, such as $\delta' = 0.010$ and $\delta' = 0.005$, the relative error is smaller than that of $\delta' = \delta = 0.05$. Therefore, according to the results shown in Table 3, as long as δ′ in the range of [0.01δ,2δ], the relative error of Scheme-2 is less than that of the finite difference method, namely Scheme-3.

3.4. Schemes comparison with missing boundary condition

When the vertical velocity boundary conditions at $z = 0$ and $z = H$ are unknown, the vertical velocity can not

Table 1
Relative error of vertical velocity at different grid resolution with δ = 0.05

	40 × 40 × 40	80 × 80 × 40	120 × 120 × 40	160 × 160 × 40
Scheme-1	3.2%	6.0%	8.8%	11.6%
Scheme-2	2.1%	1.9%	1.8%	1.8%
Scheme-3	14.1%	15.6%	17.7%	20.2%
Scheme-4	13.5%	13.5%	13.5%	13.5%

Table 2
Relative errors of Scheme-2 with different δ′

	40 × 40 × 40	80 × 80 × 40	120 × 120 × 40	160 × 160 × 40
δ′ = 0.000	7.76%	16.33%	25.30%	34.88%
δ′ = 0.0005	7.04%	7.85%	5.81%	5.01%
δ′ = 0.001	6.01%	4.31%	3.44%	3.19%
δ′ = 0.005	2.03%	1.55%	1.38%	1.27%
δ′ = 0.010	1.45%	1.10%	1.00%	0.93%
δ′ = δ = 0.05	2.10%	1.90%	1.80%	1.80%
δ′ = 0.100	6.32%	6.14%	6.11%	6.11%
δ′ = 0.250	28.09%	28.18%	28.21%	28.24%
δ′ = 0.500	60.72%	60.92%	61.00%	61.04%

Table 3
Relative errors for missing boundary condition cases with δ = 0.05

	40 × 40 × 50	80 × 80 × 50	120 × 120 × 50	160 × 160 × 50
Scheme-4	12.3%	12.3%	12.3%	12.3%
Scheme-5	71.0%	75.9%	77.6%	78.4%
Scheme-6	6.2%	6.4%	6.5%	6.5%

Table 4
Relative error of vertical velocity at different grid resolution with δ = 0.0

	40 × 40 × 40	80 × 80 × 40	120 × 120 × 40	160 × 160 × 40
Scheme-1	0.62%	0.31%	0.25%	0.23%
Scheme-2	0.55%	0.30%	0.25%	0.23%
Scheme-3	13.56%	13.59%	13.60%	13.60%
Scheme-4	13.57%	13.59%	13.60%	13.60%

be obtained by any of the four schemes. When one of the boundary conditions is unknown, the vertical velocity can be obtained by Scheme-3 and Scheme-4 which are based on the finite difference method, but can not be obtained by Scheme-1 and Scheme-2 which are based on the variational method. In many practical problems, it is difficult to obtain the top and bottom vertical velocity boundary conditions at the same time. Therefore, it is necessary to solve the problem of missing boundary conditions when Scheme-1 or Scheme-2 is used to get the vertical velocity. We assume that the vertical velocity at $z = 0$ is known and the vertical velocity at $z = H$ is unknown. Then, the following two schemes can be used to solve the problem.

Scheme-5: Firstly, assuming the vertical velocity at $z = H$ is zero, and then calculate the vertical velocity using Scheme-2.

Scheme-6: Firstly, the vertical velocity is calculated by Scheme-4, and the obtained vertical velocity at $z = H$ is taken as the boundary condition. Then, the vertical velocity is recalculated by Scheme-2.

Let $\delta = 0.05$, $x \in [0,2\pi]$, $y \in [0,2\pi]$ and $z \in [0,0.25\pi]$. According to Eq. (23) it can be seen that the vertical velocity at $z = 0.25\pi$ is not zero. Table 3 shows the relative errors calculated at different grid resolutions. As shown in the table, the relative error of Scheme-5 is very large, which indicates that when the guessed boundary condition deviates greatly from the exact boundary condition, it will bring great error to the final result.

The relative error of Scheme-6 is about 50% lower than that of Scheme-4, which shows that even if the boundary condition is predicted by the finite difference method, the variational-based method is still better than the pure finite difference method, namely Scheme-4.

3.5. Schemes comparison with δ = 0.0

To investigate the calculation results of the four schemes in the case of no observation error, the observed horizontal flow field $(\tilde{u},\tilde{v})$ is reconstructed with $\delta = 0.0$. The regularization parameter is taken as $\alpha = 0$. Table 4 shows the relative errors calculated at different grid resolutions. As shown in the table, the relative error of Scheme-3 is almost the same as that of Scheme-4 at any grid resolution, and the relative error of Scheme-1 and Scheme-2 is almost the same when the grid resolution is relatively high. The results show that the derivative calculation method has little effect on the relative error of the vertical velocity when there is no observation error.

The relative errors of Scheme-1 and Scheme-2 are obviously smaller than those of Scheme-3 and Scheme-4 at any grid resolution. When grid resolution is 160 × 160 × 40, the relative error of Scheme-1 and Scheme-2 is 98% less than that of Scheme-3 and Scheme-4. It is shown that the variational-based vertical velocity calculation method has a more obvious advantage when there is no observation error.

4. Conclusion

In this paper, a novel method for calculating the vertical velocity of the incompressible flow based on the variational method is proposed. The one-dimensional numerical differential method based on the Tikhonov regularization is used to calculate the partial derivatives. By comparison with the traditional finite difference method, we can draw the following conclusions:

- When the calculation method of the derivative is the same, the variational-based vertical velocity calculation method is superior to the traditional finite difference method.
- When the observation error exists in the horizontal flow field, the relative error of the vertical velocity can be effectively reduced by the one-dimensional numerical differential method.
- When the magnitude of the observation error is unknown, as long as the ratio between the guessed and the exact value is between 0.01–2, the vertical velocity calculated by the new method is still more accurate than that of the finite difference method.
- When the top or bottom boundary condition of the vertical velocity is absent, the missing boundary condition can be obtained by the one-dimensional numerical

differential method and the finite difference method approximately. The relative error of the vertical velocity which is re-calculated by the variational-based method decreases about 50% compared with the simple finite difference method.

In summary, the proposed vertical velocity calculation method is obviously superior to the traditional finite difference method. When the grid resolution is relatively high, the relative error can be reduced by more than 90%. This method can be widely applied to the calculation of the vertical velocity in ocean, meteorology and fluid experiment observations.

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References

[1] Z.Q. Hu, W.M. Xia, C.N. Yang, Calculating air vertical velocity using the modified EVAD technique and the variational method, J. Nanjing Inst. Meteorol., 28 (2005) 344–350.
[2] S.P. Dong, Prospective image algorithm of particle positioning and determinability of particle's position in 3-D PIV, J. Exp. Mech., 15 (2000) 163–174.
[3] J. Hu, J. Gan, Z. Sun, Observed the three-dimensional structure of a cold eddy in the southwestern South China Sea, J. Geophys. Res. Oceans, 116 (2011) 86–86.
[4] M. Perlo Cohen, I. Zamora Saenz, Environmental perspectives on the contamination and recovery of the Magdalena River in the city of Mexico, Rev. Int. Contam. Ambient, 33 (2017) 377–391.
[5] M.E. Munoz-Colmenares, S.S.S. Sarma, S. Nandini, Seasonal variations of rotifers from the high altitude llano reservoir (state of Mexico, Mexico), J. Environ. Biol., 38 (2017) 1171–1181.
[6] S. Lu, S.V. Pereverzev, Numerical differentiation from a viewpoint of regularization theory, Math. Comput., 75 (2006) 1853–1870.
[7] H.W. Engl, M. Hanke, A. Neubauer, Regularization of Inverse Problems, Kluwer Academic Publishers 2000.
[8] C.W. Groetsch, Lanczo's generalized derivative, Am. Math. Mon., 105 (1998) 320–326.
[9] C.W. Groetsch, Optimal order of accuracy in Vasin's method for differentiation of noisy functions, J. Optim. Theory Appl., 74 (1992) 373–378.
[10] C.W. Groetsch, O. Scherzer, The optimal order of convergence for stable evaluation of differential operators, Electron. J. Differ. Equations, 4 (1993) 1–10.
[11] D.A. Murio, Automatic numerical differentiation by discrete mollification, Comput. Math. Appl., 13 (1987) 381–386.
[12] J. Locker, P.M. Prenter, Regularization with differential operators, II: weak least squares finite element solutions to first kind integral equations, SIAM J. Numer. Anal., 17 (1980) 247–267.
[13] J. Locker, P.M. Prenter, Regularization with differential operators. I. General theory, J. Math. Anal. Appl., 74 (1980) 504–529.
[14] R. Anderssen, M. Hegland, For numerical differentiation, dimensionality can be a blessing!, Math. Comput., 68 (1999) 1121–1141.
[15] Y. Servati, S.H. Ghodsypour, M.A. Shirazi, Robust technology transfer policymaking using scenario-based fuzzy topics - a case study of Iran's gas industry, Appl. Ecol. Environ. Res., 15 (2017) 593–610.
[16] N.A. Abdullah, F. Rahim, Distinctiveness and potentials of two flowering roadside hedgerows, *Turnera ulmifolia* and *Melastoma malabathricum* as beneficial plants for insects, Environ. Ecosyst. Sci., 2 (2018) 6–10.
[17] P.K. Rai, A. Rai, S. Singh, Change in soil microbial biomass along a rural-urban gradient in Varanasi (UP, India), Geol. Ecol. Landscapes, 2 (2018) 15–21.
[18] M.H. Mahtab, M. Ohara, M. Rasmy, The impact of rainfall variations on flash flooding in haor areas in Bangladesh, Water Conserv. Manage., 2 (2018) 06–10.
[19] V. Havrysh, V. Nitsenko, Y. Bilan, Assessment of optimal location for a centralized biogas upgrading facility, Energy Environ., 30 (2019) 462–480.
[20] Y.B. Wang, T. Wei, Numerical differentiation for two-dimensional scattered data, J. Math. Anal. Appl., 312 (2005) 121–137.
[21] S. Hossain, Seismic geomorphology as a tool for reservoir characterization: a case study from moragot field of Pattani Basin, Gulf of Thailand, Malaysian J. Geosci., 3 (2019) 45–50.
[22] Y.B. Wang, Y.C. Hon, J. Cheng, Reconstruction of high order derivatives from input data, J. Inverse Ill-Posed Prob., 14 (2006) 205–218.
[23] N. Ahmed, M.N. Islam, M.F. Hasan, T. Motahar, M. Sujauddin, Understanding the political ecology of forced migration and deforestation through a multi-algorithm classification approach: the case of Rohingya displacement in the southeastern border region of Bangladesh, Geol. Ecol. Landscapes, 3 (2019) 282–294.
[24] Y.G. Wang, Q.F. Cai, S.X. Huang, A new method for calculating the derivation of meteorological observational data, Acta Phys. Sin. Chin. Ed., 59 (2010) 4359–436.
[25] Q.F. Cai, S.X. Huang, S.T. Gao, A new method for calculating vorticity, Acta Phys. Sin. Chin. Ed., 57 (2008) 3912–3919.
[26] Y.B. Wang, Numerical Differential and its Applications, Fudan University, 2005.
[27] N.F.H. Nordin, T.S.H. Tg Mahassan, A.J.K. Chowdhury, Bacterial population study in oil-contaminated and uncontaminated soils, J. Clean Was, 2 (2018) 06–09.