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Analytical solutions of advection–dispersion equation using fuzzy theory

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ABSTRACT

The aim of this research is to apply mathematical models for the solution of the one-dimensional convective–dispersive solute transport equation. For the derivation of the advection–dispersion equation it is assumed that the flow in the medium is unidirectional and that the average velocity is constant throughout the length of the flow field. Moreover, it is assumed that the porous medium is homogeneous and isotropic and that no mass transfer occurs between the solid and liquid phases. Unfortunately, the boundary conditions of the problem are not always intuitively apparent and in many cases convey uncertainties. For that reason, the problem is solved by utilizing fuzzy systems and fuzzy logic. The significance and the main advantage of this research is the introduction of fuzzy logic in order to solve similar problems presenting uncertainties. Since the aforementioned problem involves differential equations, a method of generalized Hukuhara derivative for total derivatives was applied, as well as the extension of the corresponding theory, concerning partial derivatives. So the fuzzy problem was transformed in a system of two classical differential equations, which were resolved with a Laplace transformation. The development of the fuzzy concentration profile is presented, as well as the membership functions of concentration. The results have given some beneficial conclusions for the effects of the uncertainties. In conclusion, it is expected that this conception will help the researchers and the engineers to take the right decision in similar problems. It is a special effort in this research to solve an advection–dispersion equation presenting uncertainties in boundaries and to follow the effect of these uncertainties in time.

Keywords: Advection–dispersion; Homogeneous porous medium; Fuzzy system; Partial differential equation; Concentration profiles

1. Introduction

Dispersion problems have been a subject studied by many investigators who are concerned with chemical constituents moving through soil by various transport mechanisms. These mechanisms act simultaneously and incorporate processes such as convection, diffusion, and dispersion. As the pollution is spreading to the subsurface environment, the advection–diffusion problem has drawn the attention of many sciences like hydrology, civil engineering, soil physics, petroleum engineering, chemical engineering, and biosciences.

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Advection–dispersion through a medium is described by a partial differential equation of parabolic type and it is widely known as the solute transport equation. While, numerical solutions are often used in such problems, many times analytical solutions are also employed, giving a better understanding of the transport mechanisms as well as estimating model parameters with inverse methods. The advection–dispersion equation of pollution incorporates the aforementioned transport mechanisms for the conservation of suspended materials. Initially, the governing equation was one-dimensional with a set of initial and boundary conditions and has been solved considering uniform dispersion and velocity, with terms accounting for linear equilibrium adsorption, zero-order production and first-order decay. In order to obtain analytical solutions, researchers have tried to reduce the advection–diffusion equation into a diffusion equation, eliminating the convective term [1–3], and consequently used the Laplace transformation technique to obtain the desired solutions. Apart from these pioneers, many others have developed numerous analytical solutions to describe the aforementioned one-dimensional convective–dispersive solute transport [4–15]. Some one-dimensional analytical solutions have been provided that approach better real problems, by transforming the non-linear advection–diffusion equation into a linear one for specific forms of the moisture content and hydraulic conductivity vs. pressure head [16]. Problems have been also presented, with variable coefficients in a finite domain [17], with temporally dependent coefficients [18], for varying pulse-type input point source [19], using the variational iteration method and the homotopy perturbation method [20], and with a sine profile for the initial condition [21].

Since the described problem concerns differential equations, which present particular problems regarding fuzzy logic, it should be mentioned that a number of studies have been already carried out in that field, especially regarding the fuzzy differentiation of functions. Initially, fuzzy differentiable functions were studied by [22], who generalized and extended Hukuhara's study [23] (H-derivative) of a set of values appearing in fuzzy sets. A theory on fuzzy differential equations is developed by [24,25]. Many studies have been carried out during the last years in the theoretical and applied research field on fuzzy differential equations with an H-derivative [26–28]. Nevertheless, in many cases, this method has presented certain drawbacks, since it has led to solutions with increasing support, along with increasing time [29,30]. This proves that, in some cases, this solution is not a good generalization of the classic case. To overcome this drawback, the generalized derivative generalized Hukuhara (gH) was introduced [31–34]. The generalized derivative gH will be used from now on for a more extensive degree of fuzzy functions than the Hukuhara derivative.

This publication concerns mathematical models for the solution of the fuzzy one-dimensional convective–dispersive solute transport equation. For the derivation of the advection–dispersion equation, it is assumed that the flow in the medium is unidirectional and the average velocity is taken to be constant throughout the length of the flow field. Besides, it is assumed that the porous medium is homogeneous and isotropic and that no mass transfer occurs between the solid and liquid phases. Unfortunately, the boundary conditions of the problem are not always intuitively evident. The uncertainty over them creates ambiguities to the solution of the problem. The hydraulic parameters of this problem are considered crisp as well as the geometric parameters. The fuzzy problem can be translated into a system of crisp boundary value problems, hereafter called the corresponding system for the fuzzy problem. Subsequently, the crisp problem is solved, the results are given in diagrams, and numerical examples are presented. The article has the following structure: firstly, the problem is presented, followed by the development of the mathematical model formulating certain characteristics regarding generalized fuzzy derivatives. Subsequently, the model is analyzed in its fuzzy form and its applications follow. Finally, the conclusions are drawn. The significance and the main advantage of this research is the introduction of fuzzy logic in order to solve problems with partial differential equations containing uncertainties. In most of the current applications of fuzzy logic in industrial systems and consumer products, a small subset of fuzzy logic is used centering on the methodology of fuzzy rules and their induction from observations. This new conception concerning partial differential equations will help the researchers and the engineers to take the right decision in similar fuzzy problems.

2. Mathematical model

2.1. Crisp case

The partial differential equation describing one-dimensional solute transport through a homogeneous medium is as follows:

$$R\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} - u\frac{\partial C}{\partial x}, \tag{1}$$

where C is the solute concentration (ML^{-3}), D is the dispersion coefficient ($L^2\ T^{-1}$), u is the pore water velocity (LT^{-1}), x is the distance, and t is the time. The retardation factor is equal to $R = 1 + \rho K_d$, where ρ is the porous media density (ML^{-3}) and K_d is the distribution coefficient ($M^{-1}\ L^3$). The retardation factor indicates that the model is a linear and reversible equilibrium adsorption. It is assumed here that K_d is negligible and $R = 1$. The initial and boundary conditions are as following:

2.1.1. Initial conditions

$$C(x,0) = 0 \tag{2}$$

2.1.2. Boundary conditions

$$C(0,t) = \begin{cases} C_0 & 0 < t < t_0, \\ 0 & t > t_0, \end{cases} \qquad \frac{\partial C}{\partial x}\Big|_{x\to\infty} = 0. \tag{3}$$

The solution of the above equation for these initial and boundary conditions is given by [1,2] as follows:

$$C(x,t)=\begin{cases} C_0F(x,t) & 0<t<t_0, \\ C_0F(x,t)-C_0F(x,t-t_0) & t>t_0, \end{cases}$$

where (4)

$$F(x,t)=\frac{1}{2}\left(\operatorname{erfc}\left(\frac{x-ut}{2\sqrt{Dt}}\right)+e^{\frac{ux}{D}}\operatorname{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right)\right).$$

2.2. Fuzzy case

2.2.1. Definitions

2.2.1.1. Definition

A fuzzy set $\tilde{C}$ on a universe set X is a mapping $\tilde{C}:X\to[0,1]$, assigning to each element $x\in X$ a degree of membership $0\le\tilde{C}(x)\le 1$. The membership function $\tilde{C}(x)$ is also defined as $\mu_{\tilde{C}}(x)$ with the properties: (i) $\mu_{\tilde{C}}(x)$ is upper semi continuous, (ii) $\mu_{\tilde{C}}(x)=0$, outside of some interval $[c,d]$, (iii) there are real numbers $c\le a\le b\le d$, such that $\mu_{\tilde{C}}$ is monotonic non-decreasing on $[c,a]$, monotonic non-increasing on $[b,d]$ and $\mu_{\tilde{C}}(x)=1$ for each $x\in[a,b]$. (iv) $\tilde{C}$ is a convex fuzzy set: $\mu_{\tilde{C}}\left(\lambda x+(1-\lambda)x\right)\ge\min\left\{\mu_{\tilde{C}}(\lambda x),\mu_{\tilde{C}}\left((1-\lambda)x\right)\right\}$.

2.2.1.2. Definition

Let X be a Banach space and $\tilde{C}$ be a fuzzy set on X. We define the α-cuts of $\tilde{C}$ as

$$\left[\tilde{C}(x)\right]^{\alpha}=\begin{cases} x\in X\middle|\tilde{C}(x)\ge\alpha, & \alpha\in(0,1] \\ \operatorname{cl}\left(\operatorname{supp}\left(\tilde{C}(x)\right)\right), & \alpha=0 \end{cases}$$

where $\operatorname{cl}\left(\operatorname{supp}\left(\tilde{C}(x)\right)\right)$ denotes the closure of the support of $\tilde{C}(x)$.

2.2.1.3. Definition

Let $[\tilde{C}]\in\mathbb{R}_f$, where $\mathbb{R}_f$ is the space of all compact and convex fuzzy sets on X. The α-cuts of $\tilde{C}$, are: $\left[\tilde{C}\right]^{\alpha}=\left[C_{\alpha}^{-},C_{\alpha}^{+}\right]$. According to representation theorem of [35] and the theorem of [36], the membership function and the α-cut form of a fuzzy number $\tilde{C}$, are equivalent and in particular the α-cuts $\left[\tilde{C}\right]_{\alpha}=\left[C_{\alpha}^{-},C_{\alpha}^{+}\right]$ uniquely represent $\tilde{C}$, provided that the two functions are monotonic (C_{α}^{-} monotonic non-decreasing, C_{α}^{+} monotonic non-increasing) and $C_{\alpha=1}^{-}\le C_{\alpha=1}^{+}$.

2.2.1.4. Definition

gH-differentiability [37]: let $\tilde{C}:[a,b]\to\mathbb{R}_f$ be such that $\left[\tilde{C}(x)\right]_{\alpha}=\left[C_{\alpha}^{-}(x),C_{\alpha}^{+}(x)\right]$. Suppose that the functions $C_{\alpha}^{-}(x)$ and $C_{\alpha}^{+}(x)$ are real-valued functions, differentiable w.r.t. x, uniformly w.r.t. $\alpha\in[0,1]$. Then the function $\tilde{C}$ is gH-differentiable at a fixed $x\in[a,b]$ if and only if one of the following two cases holds:

- $(C_{\alpha}^{-})'(x)$ is increasing, $(C_{\alpha}^{+})'(x)$ is decreasing as functions of α, and $(C_{\alpha=1}^{-})'(x)\le(C_{\alpha=1}^{+})'(x)$], or
- $(C_{\alpha}^{+})'(x)$ is increasing, $(C_{\alpha}^{-})'(x)$ is decreasing as functions of α, and $(C_{\alpha=1}^{+})'(x)\le(C_{\alpha=1}^{-})'(x)$].

Notation 1: $\left(C_{\alpha}^{-}\right)'(x)=\dfrac{dC_{\alpha}^{-}(x)}{dx}$, $\left(C_{\alpha}^{+}\right)'(x)=\dfrac{dC_{\alpha}^{+}(x)}{dx}$. In both of the above cases, $\tilde{C}'_{\alpha}(x)$ derivative is a fuzzy number.

Notation 2: the first case concerns the Hukuhara differentiability.

2.2.1.5. Definition

gH-differentiable at x_0: let $\tilde{C}:[a,b]\to\mathbb{R}_f$ and $x_0\in[a,b]$ with $C_{\alpha}^{-}(x)$ and $C_{\alpha}^{+}(x)$ both differentiable at x_0. We say that [37]:

- $\tilde{C}$ is (i)-gH-differentiable at x_0 if

$$(i)\left[\tilde{C}'_{\text{gH}}(x_0)\right]_{\alpha}=\left[(C_{\alpha}^{-})'(x_0),(C_{\alpha}^{+})'(x_0)\right],\ \forall\alpha\in[0,1] \tag{5}$$

- $\tilde{C}$ is (ii)-gH-differentiable at x_0 if

$$(ii)\ \left[\tilde{C}'_{\text{gH}}(x_0)\right]_{\alpha}=\left[(C_{\alpha}^{+})'(x_0),(C_{\alpha}^{-})'(x_0)\right],\ \forall\alpha\in[0,1] \tag{6}$$

2.2.1.6. Definition

g-differentiability: let $\tilde{C}:[a,b]\to\mathbb{R}_f$ be such that $\left[\tilde{C}(x)\right]_{\alpha}=\left[C_{\alpha}^{-}(x),C_{\alpha}^{+}(x)\right]$. If $C_{\alpha}^{-}(x)$ and $C_{\alpha}^{+}(x)$ are differentiable real-valued functions with respect to x, uniformly for $\alpha\in[0,1]$, then $\tilde{C}(x)$ is g-differentiable and we have [37]:

$$\left[\tilde{C}'_{g}(x)\right]_{\alpha}=\begin{bmatrix}\inf\limits_{\beta\ge\alpha}\min\left\{\left(C_{\alpha}^{-}\right)'(x),\left(C_{\alpha}^{+}\right)'(x)\right\}, \\ \sup\limits_{\beta\ge\alpha}\max\left\{\left(C_{\alpha}^{-}\right)'(x),\left(C_{\alpha}^{+}\right)'(x)\right\}\end{bmatrix}. \tag{7}$$

2.2.1.7. Definition

The gH-differentiability implies g-differentiability, but the inverse is not true.

2.2.1.8. Definition

[gH-p] differentiability: a fuzzy-valued function $\tilde{C}$ of two variables is a rule that assigns to each ordered pair of real numbers (x,t) in a set $\mathbb{D}$, a unique fuzzy number denoted by $\tilde{C}(x,t)$. Let $\tilde{C}(x,t)$: $\mathbb{D}\in\mathbb{R}_f$, $(x_0,t_0)\in\mathbb{D}$ and $C_{\alpha}^{-}(x,t)$, $C_{\alpha}^{+}(x,t)$ being real valued functions and partial differentiable w.r.t. x. We say that [34,38,39]:

$\tilde{C}$ is [(i)-p]-differentiable w.r.t. x at (x_0,t_0) if:

$$\frac{\partial\tilde{C}_{\alpha}(x_0,t_0)}{\partial x_{i,\text{gH}}}=\left[\frac{\partial C_{\alpha}^{-}(x_0,t_0)}{\partial x},\frac{\partial C_{\alpha}^{+}(x_0,t_0)}{\partial x}\right] \tag{8}$$

$\tilde{C}$ is [(ii)-p]-differentiable w.r.t. x at (x_0,t_0) if:

$$\frac{\partial\tilde{C}_{\alpha}(x_0,t_0)}{\partial x_{i,\text{gH}}}=\left[\frac{\partial C_{\alpha}^{+}(x_0,t_0)}{\partial x},\frac{\partial C_{\alpha}^{-}(x_0,t_0)}{\partial x}\right] \tag{9}$$

Notation: the same is valid for $\frac{\partial \tilde{C}_\alpha(x_0,t_0)}{\partial t}$.

2.2.1.9. Definition

Let $\tilde{C}(x,t)$: $\mathbb{D} \in \mathbb{R}_f$, and $\frac{\partial \tilde{C}_\alpha(x_0,t_0)}{\partial x_{i,\text{gH}}}$ be [gH-p]-differentiable at $(x_0,t_0) \in \mathbb{D}$ with respect to x. We say that [34,38]:

- $\frac{\partial \tilde{C}_\alpha(x_0,t_0)}{\partial x_{i,\text{gH}}}$ is [(*i*)-*p*]-differentiable w.r.t. x if:

$$\frac{\partial^2 \tilde{C}_\alpha(x_0,t_0)}{\partial x^2_{i,\text{gH}}} = \begin{cases} \left[\frac{\partial^2 C^-_\alpha(x_0,t_0)}{\partial x^2}, \frac{\partial^2 C^+_\alpha(x_0,t_0)}{\partial x^2}\right] & \text{if } \tilde{C}(x,t) \text{ is } [(i)-p] \\ \text{differentiable} & \\ \left[\frac{\partial^2 C^+_\alpha(x_0,t_0)}{\partial x^2}, \frac{\partial^2 C^-_\alpha(x_0,t_0)}{\partial x^2}\right] & \text{if } \tilde{C}(x,t) \text{ is } [(ii)-p] \\ \text{differentiable} & \end{cases} \tag{10}$$

- $\frac{\partial \tilde{C}_\alpha(x_0,t_0)}{\partial x_{ii,\text{gH}}}$ is [(*ii*)-*p*]-differentiable w.r.t. x if:

$$\frac{\partial^2 \tilde{C}_\alpha(x_0,t_0)}{\partial x^2_{ii,\text{gH}}} = \begin{cases} \left[\frac{\partial^2 C^+_\alpha(x_0,t_0)}{\partial x^2}, \frac{\partial^2 C^-_\alpha(x_0,t_0)}{\partial x^2}\right] \\ \text{if } \tilde{C}(x,t) \text{ is } [(i)-p] \text{ differentiable} \\ \left[\frac{\partial^2 C^-_\alpha(x_0,t_0)}{\partial x^2}, \frac{\partial^2 C^+_\alpha(x_0,t_0)}{\partial x^2}\right] \\ \text{if } \tilde{C}(x,t) \text{ is } [(ii)-p] \text{ differentiable} \end{cases} \tag{11}$$

2.2.2. Fuzzy model

Eq. (1) in its fuzzy form becomes:

$$\frac{\partial \tilde{C}}{\partial t} = D\frac{\partial^2 \tilde{C}}{\partial x^2} - u\frac{\partial \tilde{C}}{\partial x} \tag{12}$$

with the new initial and boundary conditions:

$$\tilde{C}(x,0) = \tilde{0},$$
$$\left[\tilde{C}(0,t)\right]_\alpha = C_0\left[1-r(1-\alpha),\ 1+r(1-\alpha)\right],\ t>0,\ \ \frac{\partial \tilde{C}(x,t)}{\partial x}\Big|_{x\to\infty} = \tilde{0}. \tag{13}$$

Fig. 1 illustrates the boundary condition $[C(0,t)]\alpha$. Solutions to the fuzzy problem Eq. (12) and the initial and boundary conditions Eq. (13) can be obtained, utilizing the theory of [34,37,38,40,41], by translating the above fuzzy problem to a system of second-order of crisp boundary value problems, called the corresponding system for the fuzzy problem. Therefore, eight crisp boundary value problems systems are possible for the fuzzy problem {(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4)}.

(1,1) system

$$\frac{\partial C^-}{\partial t} = D\frac{\partial^2 C^-}{\partial x^2} - u\frac{\partial C^-}{\partial x}$$
$$\frac{\partial C^+}{\partial t} = D\frac{\partial^2 C^+}{\partial x^2} - u\frac{\partial C^+}{\partial x}$$

(1,2) system

$$\frac{\partial C^-}{\partial t} = D\frac{\partial^2 C^-}{\partial x^2} - u\frac{\partial C^+}{\partial x}$$
$$\frac{\partial C^+}{\partial t} = D\frac{\partial^2 C^+}{\partial x^2} - u\frac{\partial C^-}{\partial x}$$

(1,3) system

$$\frac{\partial C^-}{\partial t} = D\frac{\partial^2 C^+}{\partial x^2} - u\frac{\partial C^+}{\partial x}$$
$$\frac{\partial C^+}{\partial t} = D\frac{\partial^2 C^-}{\partial x^2} - u\frac{\partial C^-}{\partial x}$$

(1,4) system

$$\frac{\partial C^-}{\partial t} = D\frac{\partial^2 C^+}{\partial x^2} - u\frac{\partial C^-}{\partial x}$$
$$\frac{\partial C^+}{\partial t} = D\frac{\partial^2 C^-}{\partial x^2} - u\frac{\partial C^+}{\partial x}$$

(2,1) system

$$\frac{\partial C^+}{\partial t} = D\frac{\partial^2 C^+}{\partial x^2} - u\frac{\partial C^+}{\partial x}$$
$$\frac{\partial C^-}{\partial t} = D\frac{\partial^2 C^-}{\partial x^2} - u\frac{\partial C^-}{\partial x}$$

(2,2) system

$$\frac{\partial C^+}{\partial t} = D\frac{\partial^2 C^+}{\partial x^2} - u\frac{\partial C^-}{\partial x}$$
$$\frac{\partial C^-}{\partial t} = D\frac{\partial^2 C^-}{\partial x^2} - u\frac{\partial C^+}{\partial x}$$

(2,3) system

$$\frac{\partial C^+}{\partial t} = D\frac{\partial^2 C^-}{\partial x^2} - u\frac{\partial C^-}{\partial x}$$
$$\frac{\partial C^-}{\partial t} = D\frac{\partial^2 C^+}{\partial x^2} - u\frac{\partial C^+}{\partial x}$$

(2,4) system

$$\frac{\partial C^+}{\partial t} = D\frac{\partial^2 C^-}{\partial x^2} - u\frac{\partial C^+}{\partial x}$$
$$\frac{\partial C^-}{\partial t} = D\frac{\partial^2 C^+}{\partial x^2} - u\frac{\partial C^-}{\partial x}$$

Subsequently, the solution of the (1,1) system, is described in detail.

$$\begin{aligned} &\frac{\partial C^-}{\partial t} = D\frac{\partial^2 C^-}{\partial x^2} - u\frac{\partial C^-}{\partial x} \\ &\quad C^-(x,0)=0,\ C^-(0,t)=C_0(1-r(1-\alpha)),\ \frac{\partial C^-(\infty,t)}{\partial x}=0, \\ &\frac{\partial C^+}{\partial t} = D\frac{\partial^2 C^+}{\partial x^2} - u\frac{\partial C^+}{\partial x} \\ &\quad C^+(x,0)=0,\ C^+(0,t)=C_0(1+r(1-\alpha)),\ \frac{\partial C^+(\infty,t)}{\partial x}=0, \end{aligned} \tag{14}$$

2.2.2.1. Solution of the system (1,1)

First case:

$$\frac{\partial C^-}{\partial t} = D\frac{\partial^2 C^-}{\partial x^2} - u\frac{\partial C^-}{\partial x} \tag{15}$$

Boundary conditions:

$$\begin{aligned} &C^-(0,t) = C_0(1-r(1-\alpha))\ t>0, \\ &\frac{\partial C^-(\infty,t)}{\partial x} = 0\ t \ge 0, \end{aligned} \tag{16}$$

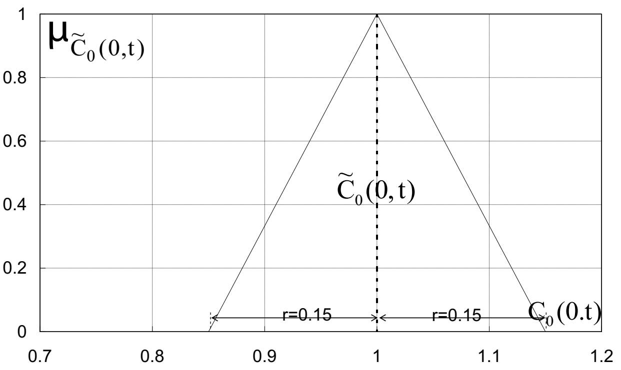

Fig. 1. Membership function of $\tilde{C}(0,t)$.

Initial conditions:

$$C^{-}\left(x,0\right)=0 \quad x\geq 0. \tag{17}$$

By setting $F = C^{-}$ in Eq. (15) we obtain the following Laplace transformation:

$$\mathcal{L}\left\{D\frac{\partial^2 F}{\partial x^2}-u\frac{\partial F}{\partial x}-\frac{\partial F}{\partial t}\right\}=D\frac{d^2\bar{F}}{dx^2}-u\frac{d\bar{F}}{dx}=s\bar{F} \tag{18}$$

with boundary conditions:

$$\bar{F}\left(0,s\right)=\frac{C_0k_1}{s},\ k_1=1-r\left(1-\alpha\right)$$
$$\frac{\partial\bar{F}\left(\infty,s\right)}{\partial x}=0. \tag{19}$$

The solution of Eq. (18) becomes [12,13]:

$$\bar{F}\left(x,s\right)=A(s)\exp\left(\frac{ux}{2D}-\frac{x}{\sqrt{D}}\sqrt{\frac{u^2}{4D}+s}\right)+$$
$$B(s)\exp\left(\frac{ux}{2D}+\frac{x}{\sqrt{D}}\sqrt{\frac{u^2}{4D}+s}\right) \tag{20}$$

The first derivative w.r.t. x is:

$$\frac{\partial\bar{F}\left(x,s\right)}{\partial x}=A(s)f_1(s)\exp\left(\frac{ux}{2D}-\frac{x}{\sqrt{D}}\sqrt{\frac{u^2}{4D}+s}\right)+$$
$$B(s)f_2(s)\exp\left(\frac{ux}{2D}+\frac{x}{\sqrt{D}}\sqrt{\frac{u^2}{4D}+s}\right) \tag{21}$$

The variable $B(s)$ should be equal to 0, in order to satisfy the boundary condition Eq. (19): $\frac{\partial\bar{F}(\infty,s)}{\partial x}=0.$
So Eq. (20) becomes:

$$\bar{F}(x,s)=A(s)\exp\left(\frac{ux}{2D}-\frac{x}{\sqrt{D}}\sqrt{\frac{u^2}{4D}+s}\right). \tag{22}$$

For the first condition for $x = 0$:

$$\bar{F}(0,s)=\frac{C_0k_1}{s},\ \bar{F}(0,s)=A(s)=\frac{C_0k_1}{s} \tag{23}$$

and Eq. (22) becomes:

$$\bar{F}(x,s)=\frac{C_0k_1}{s}\exp\left(\frac{ux}{2D}-\frac{x}{\sqrt{D}}\sqrt{\frac{u^2}{4D}+s}\right)=$$
$$C_0k_1\exp\left(\frac{ux}{2D}\right)\frac{1}{s}\exp\left(-\frac{x}{\sqrt{D}}\sqrt{\frac{u^2}{4D}+s}\right). \tag{24}$$

Applying now the inverse Laplace transformation [12,13] to Eq. (24) the following equation is obtained:

$$F=C^{-}=\frac{C_0k_1}{2}\left[\operatorname{erfc}\left(\frac{x-ut}{2\sqrt{Dt}}\right)+e^{\frac{ux}{D}}\operatorname{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right)\right] \tag{25}$$

Second case:

$$\frac{\partial C^{+}}{\partial t}=D\frac{\partial^2 C^{+}}{\partial x^2}-u\frac{\partial C^{+}}{\partial x} \tag{26}$$

Boundary conditions:

$$C^{+}(0,t)=C_0(1+r(1-\alpha))\ t>0,$$
$$\frac{\partial C^{+}(\infty,t)}{\partial x}=0\ t\geq 0, \tag{27}$$

Initial condition: $C^{+}(x,0)=0 \quad x\geq 0.$
In Eq. (26) $G = C^{+}$ is set and the following Laplace transformation is taken:

$$\mathcal{L}\left\{D\frac{\partial^2 G}{\partial x^2}-u\frac{\partial G}{\partial x}-\frac{\partial G}{\partial t}\right\}=D\frac{d^2\bar{G}}{dx^2}-u\frac{d\bar{G}}{dx}=s\bar{G} \tag{28}$$

with boundary conditions:

$$\bar{G}(0,s)=\frac{C_0k_2}{s},\ k_2=1+r(1-\alpha)$$
$$\frac{\partial\bar{G}(\infty,s)}{\partial x}=0. \tag{29}$$

Applying the same process as in the first case:

$$G=C^{+}=\frac{C_0k_2}{2}\left[\operatorname{erfc}\left(\frac{x-ut}{2\sqrt{Dt}}\right)+e^{\frac{ux}{D}}\operatorname{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right)\right] \tag{30}$$

Finally, the fuzzy solution is:

$$\tilde{C}=\frac{C_0\tilde{A}}{2}\left[\operatorname{erfc}\left(\frac{x-ut}{2\sqrt{Dt}}\right)+e^{\frac{ux}{D}}\operatorname{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right)\right] \tag{31}$$

In Eq. (31) the fuzzy number $\tilde{A}$ is as follows:

$$\left[\tilde{A}\right]_\alpha=\left[1-r\left(1-\alpha\right),\ 1+r\left(1-\alpha\right)\right] \tag{32}$$

2.2.2.2. Fuzzy derivatives

2.2.2.2.1. First Derivative of $\tilde{C}$ vs. x

In order to find the first derivative of $\tilde{C}$ w.r.t. x, [42] is applied:

$$\frac{\partial\tilde{C}}{\partial x}=\frac{C_0\tilde{A}}{2}\frac{\partial}{\partial x}\left\{\left[\text{erfc}\left(\frac{x-ut}{2\sqrt{Dt}}\right)+e^{\frac{ux}{D}}\text{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right)\right]\right\} \quad (33)$$

The derivative becomes:

$$\frac{\partial\tilde{C}}{\partial x}=\frac{C_0\tilde{A}}{2}\left[-\frac{2}{\sqrt{Dt\pi}}\exp\left\{-\left(\frac{x-ut}{2\sqrt{Dt}}\right)^2\right\}+\frac{u}{D}e^{\frac{ux}{D}}\text{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right)\right] \quad (34)$$

Now: $-u\frac{\partial\tilde{C}}{\partial x}=\tilde{A}f_1(x,t)$, is set where:

$$f_1(x,t)=-u\frac{C_0}{2}\left[-\frac{2}{\sqrt{Dt\pi}}\exp\left\{-\left(\frac{x-ut}{2\sqrt{Dt}}\right)^2\right\}+\frac{u}{D}e^{\frac{ux}{D}}\text{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right)\right], \text{ or} \quad (35)$$

$$\frac{f_1(x,t)}{uC_0/2}=\frac{2}{\sqrt{Dt\pi}}\exp\left\{-\left(\frac{x-ut}{2\sqrt{Dt}}\right)^2\right\}-\frac{u}{D}e^{\frac{ux}{D}}\text{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right) \quad (36)$$

The above relationship $-u\frac{\partial\tilde{C}}{\partial x}=\tilde{A}f_1(x,t)$, is obtained according to theorem 1 of [43], in which it is mentioned: For any: λ, $\mu \in R$ and any fuzzy number $\tilde{u}$ we have: $\lambda\cdot(\mu\cdot\tilde{u})=(\lambda\cdot\mu)\cdot\tilde{u}$.

2.2.2.2.2. First derivative of C vs. t

$$\frac{\partial\tilde{C}}{\partial t}=\frac{C_0\tilde{A}}{2}\left[\frac{x}{t\sqrt{Dt\pi}}\exp\left\{-\left(\frac{x-ut}{2\sqrt{Dt}}\right)^2\right\}\right]=\tilde{A}f_2(x,t), \text{ or} \quad (37)$$

$$\frac{f_2(x,t)}{C_0/2}=\frac{x}{t\sqrt{Dt\pi}}\exp\left\{-\left(\frac{x-ut}{2\sqrt{Dt}}\right)^2\right\} \quad (38)$$

2.2.2.2.3. Second derivative of C vs. x

$$\frac{\partial^2\tilde{C}}{\partial x^2}=\frac{C_0\tilde{A}}{2}\left[\begin{array}{l}\frac{x-2ut}{Dt\sqrt{Dt\pi}}\exp\left\{-\left(\frac{x-ut}{2\sqrt{Dt}}\right)^2\right\}+\\ \left(\frac{u}{D}\right)^2e^{\frac{ux}{D}}\text{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right)\end{array}\right]=\tilde{A}f_3(x,t) \quad (39)$$

where:

$$f_3(x,t)=\frac{C_0}{2}\left[\begin{array}{l}\frac{x-2ut}{Dt\sqrt{Dt\pi}}\exp\left\{-\left(\frac{x-ut}{2\sqrt{Dt}}\right)^2\right\}+\\ \left(\frac{u}{D}\right)^2e^{\frac{ux}{D}}\text{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right)\end{array}\right], \text{ or} \quad (40)$$

$$\frac{f_3(x,t)}{C_0/2})=\frac{x-2ut}{Dt\sqrt{Dt\pi}}\exp\left\{-\left(\frac{x-ut}{2\sqrt{Dt}}\right)^2\right\}+\left(\frac{u}{D}\right)^2e^{\frac{ux}{D}}\text{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right) \quad (41)$$

2.2.2.3. Existence statement of Eq. (12)

2.2.2.3.1. Boundary conditions

The solution of Eq. (12) for system (1,1) is Eq. (31):

$$\tilde{C}=\frac{C_0\tilde{A}}{2}\left[\text{erfc}\left(\frac{x-ut}{2\sqrt{Dt}}\right)+e^{\frac{ux}{D}}\text{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right)\right].$$

This equation for $x = 0$ and t = constant becomes:

$$\tilde{C}(0,t)=\frac{C_0\tilde{A}}{2}\left[\text{erfc}\left(\frac{-ut}{2\sqrt{Dt}}\right)+\text{erfc}\left(\frac{ut}{2\sqrt{Dt}}\right)\right]=\frac{C_0\tilde{A}}{2}2=C_0\tilde{A}.$$

So Eq. (31) satisfies boundary condition for $x = 0$. The first derivative of Eq. (24) w.r.t. x is Eq. (34):

$$\frac{\partial\tilde{C}}{\partial x}=\frac{C_0\tilde{A}}{2}\left[-\frac{2}{\sqrt{Dt\pi}}\exp\left\{-\left(\frac{x-ut}{2\sqrt{Dt}}\right)^2\right\}+\frac{u}{D}e^{\frac{ux}{D}}\text{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right)\right]$$

which for $x \to \infty$ and t = constant becomes:

$$\frac{\partial\tilde{C}}{\partial x}=\frac{C_0\tilde{A}}{2}\left[\begin{array}{l}-\frac{2}{\sqrt{Dt\pi}}\exp\left\{-\left(\frac{x-ut}{2\sqrt{Dt}}\right)^2\right\}_{x\to\infty}+\\ \frac{u}{D}\left(e^{\frac{ux}{D}}\right)\Big|_{x\to\infty}\text{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right)_{x\to\infty}\end{array}\right]$$

The first member is:

$$-\frac{2}{\sqrt{Dt\pi}}\exp\left\{-\left(\frac{x-ut}{2\sqrt{Dt}}\right)^2\right\}_{|x\to\infty}=0.$$

The second member is written as:

$$\frac{u}{D}\left(e^{\frac{ux}{D}}\right)\Big|_{x\to\infty}\text{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right)_{x\to\infty}=\frac{u}{D}\frac{\text{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right)_{x\to\infty}}{\left(e^{-\frac{ux}{D}}\right)\Big|_{x\to\infty}}=\frac{u}{D}\frac{0}{0}.$$

The L'Hôpital rule is applied:

$$\frac{\left(\text{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right)\right)'_{x\to\infty}}{\left(\left(e^{-\frac{ux}{D}}\right)\right)'\Big|_{x\to\infty}}=\frac{-2/\sqrt{\pi}\exp\left(-\left(\frac{x+ut}{2\sqrt{Dt}}\right)^2\right)_{x\to\infty}}{-(u/D)e^{-\frac{ux}{D}}\Big|x\to\infty}=0 \quad .$$

Finally: $\frac{\partial \tilde{C}}{\partial x}\Big|_{x\leftarrow\infty} = 0$. So Eq. (24) satisfies the boundary condition for $x \to \infty$.

2.2.2.3.2. Initial condition

Eq. (31) for $t = 0$ and x = constant becomes:

$$\tilde{C}(x,0) = \frac{C_0\tilde{A}}{2}\left[\mathrm{erfc}\left(\frac{x}{0}\right) + e^{\frac{ux}{D}}\mathrm{erfc}\left(\frac{x}{0}\right)\right] = \frac{C_0\tilde{A}}{2}\left[\mathrm{erfc}(\infty) + e^{\frac{ux}{D}}\mathrm{erfc}(\infty)\right] = 0.$$

So Eq. (31) satisfies the initial condition.

Remark 1: Eq. (31) for $x \to \infty$ becomes:

$$\left[\tilde{C}\right]_{\alpha\atop x\to\infty} = \left[C^-,C^+\right]_{\alpha\atop x\to\infty} = \begin{bmatrix} \frac{C_0(1-r(1-\alpha)}{2}f(x,t), \\ \frac{C_0(1+r(1-\alpha)}{2}f(x,t) \end{bmatrix}_{x\to\infty} = [0,0]$$

Remark 2: Eq. (31) for $t \to \infty$ becomes:

$$\left[\tilde{C}\right]_{\alpha\atop t\to\infty} = \left[C^-,C^+\right]_{\alpha\atop t\to\infty} = \begin{bmatrix} \frac{C_0(1-r(1-\alpha)}{2}f(x,t), \\ \frac{C_0(1+r(1-\alpha)}{2}f(x,t) \end{bmatrix}_{t\to\infty} =$$

$$= \left[C_0\left(1-r(1-\alpha)\right), C_0(1+r(1-\alpha)\right)\right], \text{ or}$$

$$\frac{[\tilde{C}]_\alpha}{C_0} = \left[(1-r(1-\alpha)),(1+r(1-\alpha))\right],$$

where $f(x,t)\big|_{t\to\infty} = \left[\mathrm{erfc}\left(\frac{x-ut}{2\sqrt{Dt}}\right) + e^{\frac{ux}{D}}\mathrm{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right)\right]_{t\to\infty} =$ $\mathrm{erfc}(-\infty) + \mathrm{erfc}(\infty) = 2$.

2.2.2.4. Satisfaction of Eq. (12)

The following equation should be valid:

$$\frac{\partial \tilde{C}}{\partial t} = D\frac{\partial^2 \tilde{C}}{\partial x^2} - u\frac{\partial \tilde{C}}{\partial x}, \text{ or} \tag{42}$$

$$\tilde{A}f_2(x,t) = \tilde{A}f_1(x,t) + \tilde{A}Df_3(x,t) \tag{43}$$

In the above equation we apply to the right part of the equation the theorem 1 of [43]: For any $a,b \in R$ with $a,b \geq 0$, or $a,b \leq 0$ and any fuzzy number $\tilde{u} \in \mathbb{R}_f$ we have: $(a + b)\cdot\tilde{u} = a\cdot\tilde{u} + b\cdot\tilde{u}$. Now the above equation becomes:

$$\tilde{A}f_2(x,t) = \tilde{A}\left\{f_1(x,t) + Df_3(x,t)\right\}, \tag{44}$$

providing that the functions $f_2(x,t)$, $f_3(x,t)$ are either both positive or both negative. In the following figures the functions $f_1(x,t)$, $f_2(x,t)$, $f_3(x,t)$ are illustrated as functions of x,t.

$$\frac{f_1(x,t)}{uC_0/2} = \frac{2}{\sqrt{Dt\pi}}\exp\left\{-\left(\frac{x-ut}{2\sqrt{Dt}}\right)^2\right\} - \frac{u}{D}e^{\frac{ux}{D}}\mathrm{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right) \tag{45}$$

$$\frac{f_2(x,t)}{C_0/2} = \frac{x}{t\sqrt{Dt\pi}}\exp\left\{-(\frac{x-ut}{2\sqrt{Dt}})^2\right\} \tag{46}$$

$$\left(\frac{f_3(x,t)}{C_0/2}\right) = \frac{x-2ut}{Dt\sqrt{Dt\pi}}\exp\left\{-\left(\frac{x-ut}{2\sqrt{Dt}}\right)^2\right\} + \left(\frac{u}{D}\right)^2 e^{\frac{ux}{D}}\mathrm{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right) \tag{47}$$

$$\begin{aligned} f_1(x,t) + Df_3(x,t) &= -u\frac{C_0}{2}\begin{bmatrix} -\frac{2}{\sqrt{Dt\pi}}\exp\left\{-\left(\frac{x-ut}{2\sqrt{Dt}}\right)^2\right\} + \\ \frac{u}{D}e^{\frac{ux}{D}}\mathrm{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right) \end{bmatrix} \\ &+ \frac{DC_0}{2}\left[\frac{x-ut}{Dt\sqrt{Dt\pi}}\exp\left\{-\left(\frac{x-ut}{2\sqrt{Dt}}\right)^2\right\} + \left(\frac{u}{D}\right)^2 e^{\frac{ux}{D}}\mathrm{erfc}\left(\frac{x+ut}{2\sqrt{Dt}}\right)\right. \\ &\left. -\frac{ut}{Dt\sqrt{Dt\pi}}\exp\left\{-\left(\frac{x-ut}{2\sqrt{Dt}}\right)^2\right\}\right] = \frac{C_0}{2}\left[\frac{x}{t\sqrt{Dt\pi}}\exp\left\{-\left(\frac{x-ut}{2\sqrt{Dt}}\right)^2\right]\right. \\ &= f_2(x,t) \end{aligned} \tag{48}$$

As it can be seen from the above figures the functions $f_1(x,t)$, $f_2(x,t)$, are positive defined in $\mathbb{R}$. In order to simplify and clarify $f_3(x,t)$ we pose now non dimensional coordinates $\xi = ut/x$ and $\eta = D/ux$ and we obtain:

$$f_3(\xi,\eta) = \frac{C_0}{2\eta x^2}g_3(\eta,\xi)$$

where:

$$g_3(\eta,\xi) = \left[\frac{1-2\xi}{\xi\sqrt{\eta\xi}}\exp\left\{-\left(\frac{1-\xi}{2\sqrt{\eta\xi}}\right)^2\right\} + \frac{1}{\eta}e^{\frac{1}{\eta}}\mathrm{erfc}\left(\frac{1+\xi}{2\sqrt{\eta\xi}}\right)\right].$$

Figs. 2 and 3 illustrate the above functions $f1(x,t)/uC_0/2$ and $f2(x,t)/C_0/2$ vs. x and Fig. 4 illustrates the above function $f_3(\xi,\eta)$ vs. ξ. It is to be noted that for large values of η and $\xi \in [0,0.5]$ the above function takes positives values. For small values of η and for $\xi \in [0,1]$ the function $f_3(z,t) = f_3(\xi,\eta)$ takes also positives values. As ξ is equal to ut/x it means that there is an upper bound for the ratio $t/x \leq 0.5/u$ in the first case for large values of η and $t/x \leq 1/u$ in the second case for small values of η. Beyond these limits the function $f_3(z,t)$ takes negative values. The above limits define a linear relation between the coordinates t and x according to [44–47], and the function $f_3(z,t)$ is proved to be positive, subject to certain limits.

As can be seen from the above figures the functions $f_1(x,t)$, $f_2(x,t)$, is positively defined in $\mathbb{R}$, while the function $f_3(x,t)$ is positively defined only inside certain limits of x, t. That means that the derivatives $\frac{\partial \tilde{C}}{\partial x}, \frac{\partial \tilde{C}}{\partial t}$ are valid fuzzy numbers in $\mathbb{R}_f$ and according to definition 2.2.1.8 we have:

$$\frac{\partial \tilde{C}_\alpha(x,t)}{\partial x} = \left[\frac{\partial C_\alpha^-(x,t)}{\partial x}, \frac{\partial C_\alpha^+(x,t)}{\partial x}\right],$$

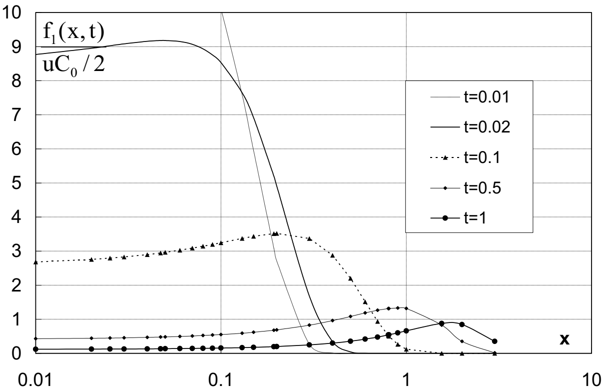

Fig. 2. Function $\frac{f_1(x,t)}{uC_0/2}$ vs. x.

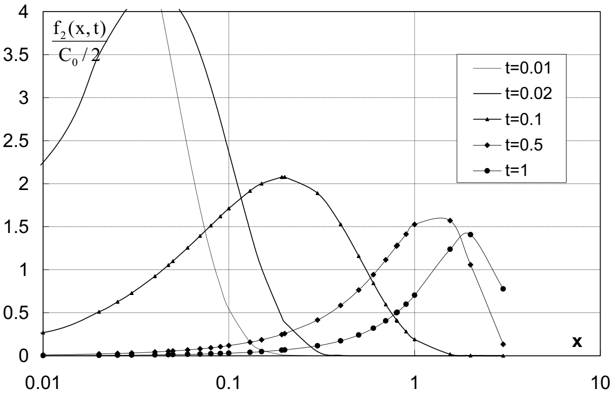

Fig. 3. Function $\frac{f_2(x,t)}{C_0/2}$ vs. x.

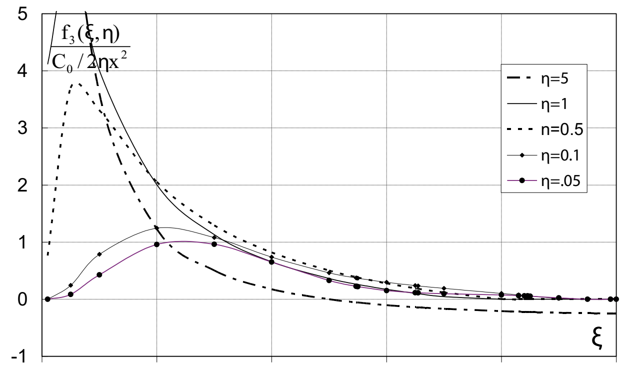

Fig. 4. Function $\frac{f_3(\xi,\eta)}{C_0/2\eta x^2}$ vs. x.

$$\frac{\partial \tilde{C}_\alpha(x,t)}{\partial t} = \left[\frac{\partial C_\alpha^-(x,t)}{\partial t}, \frac{\partial C_\alpha^+(x,t)}{\partial t}\right],$$

and $\tilde{C}(x,t)$ is (i–p) differentiable. The second derivative $\frac{\partial^2 \tilde{C}}{\partial x^2}$ is a valid fuzzy number and the function $\frac{f_3(x,t)}{C_0/2}$ is positively defined only inside certain limits of $f_3(x,t)$ and according to definition 2.2.1.9 we have:

$$\frac{\partial^2 \tilde{C}_\alpha(x,t)}{\partial x^2} = \left[\frac{\partial^2 C_\alpha^-(x,t)}{\partial x^2}, \frac{\partial^2 C_\alpha^+(x,t)}{\partial x^2}\right]$$

since $\tilde{C}(x,t)$ is (i–p) differentiable. That means that the fuzzy solution of the above problem is valid only inside certain limits.

3. Application

Two cases were examined, and the values assigned to different parameters are given in the following Table 1:

In the first case, concentration values are evaluated at $t = 0.055$, 0.274, 0.55, and 1 y or $t = 20$, 100, 200, and 365 d. Fig. 5 illustrates the time dependent solute profiles at the above times. Fig. 6 illustrates membership functions of concentration for times $t = 100$, 200, and 365 d at position $x = 1$ km. Finally, Fig. 7 illustrates the concentration vs. time at positions $x = 0.5$ and 1.5 km. The fronts of the concentration attained in Fig. 5 are the positions $x_{20d} = 0.6$ km, $x_{100d} = 1.5$ km, $x_{200d} = 3$ km, and $x_{365d} = 4$ km. As it can be observed at Fig. 6 the reduced concentration C/C_0 at time $t = 365$ d and in position $x = 1$ km attains the values 0.69, 0.82, 0.94 (at level $\alpha = 0$). Consequently, the initially uncertainty of 15% about the true value remains the same. In Fig. 7 the reduced concentration profiles vs. t, attain the value $1 \pm r$, where $r = 0.15$ is the spread.

In the second case, concentration values are evaluated at $t = 0.1$, 0.4, 0.7, and 1 y. Fig. 8 illustrates the solute profiles at the above times. Fig. 9 illustrates membership functions

Table 1
Values of parameters D and u

a/a	D (km² y⁻¹)	u (km y⁻¹)
1	0.50	1.50
2	0.21	0.11

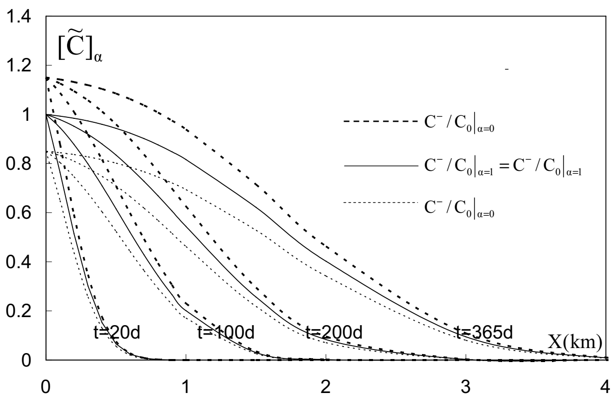

Fig. 5. Profiles of concentration for times $t = 20$, 100, 200, and 365 d.

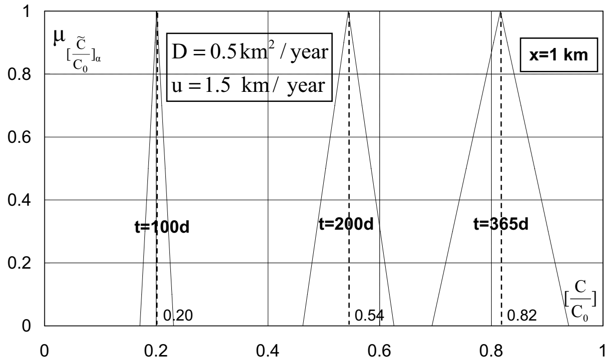

Fig. 6. Membership functions of concentration for times t = 100, 200, and 365 d.

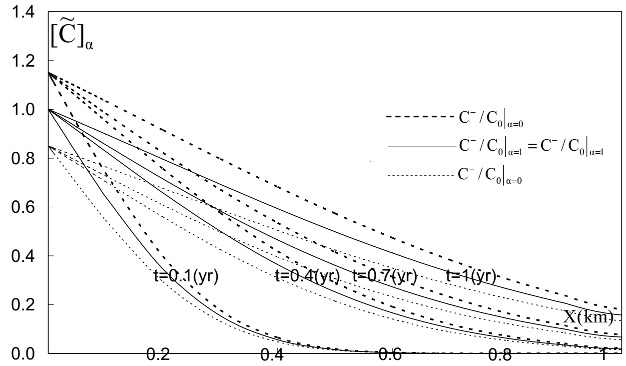

Fig. 8. Profiles of concentration for times t = 0.1, 0.4, 0.7, and 1 y.

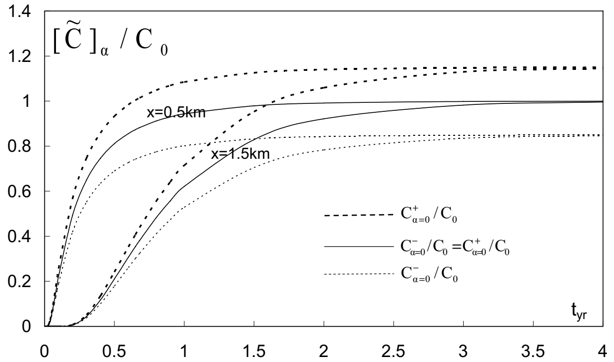

Fig. 7. Concentration vs. time at positions x = 0.5 and 1.5 km.

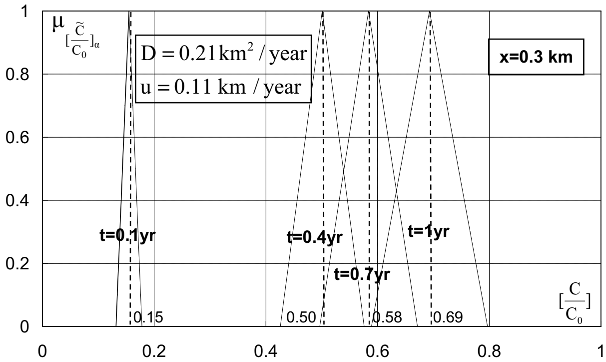

Fig. 9. Membership functions of concentration for times t = 0.1, 0.4, and 1 y.

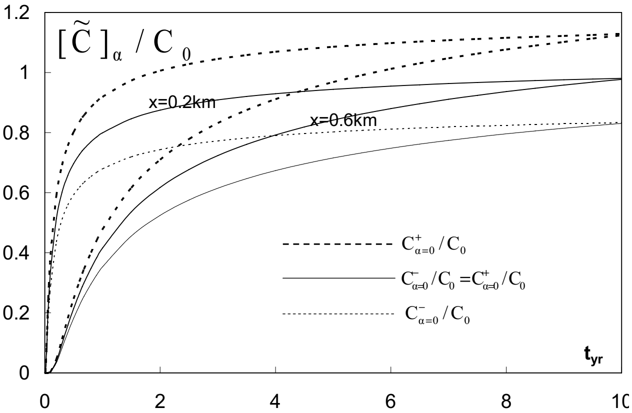

Fig. 10. Concentration vs. time at positions x = 0.2 and 0.6 km.

of concentration for times t_y = 0.1, 0.4, 0.7, and 1 at position x = 0.3 km. Finally, Fig. 10 illustrates the concentration vs. time at positions x = 0.2 km and 0.6 km. The fronts of the concentration attained in Fig. 9 are at the positions $x_{0.1y}$ = 0.15 km, $x_{0.4y}$ = 0.5 km, $x_{0.7y}$ = 0.58 km, x_{1y} = 0.69 km. As it can be observed at Fig. 9, the reduced concentration C/C_0 at time t = 1 y and in position x = 0.3 km attains the values of 0.59, 0.69 and 0.80 (at level α = 0). Consequently, the initially uncertainty of 15% about the true value remains the same. In Fig. 10 the reduced concentration profiles vs. t, attain the value 1 ± r, where r = 0.15 is the spread.

4. Conclusion

The [37] theory of the gH derivative, as well as its extension by [33] to partial differential equations, allows researchers to solve practical problems, which are useful in engineering. It is now possible for engineers consider the fuzziness of various parameters during calculations.

The advection–dispersion equation in case of the corresponding system (1,1) has a fuzzy solution with certain restrictions: The first derivative with respect to x, as well as the first derivative with respect to t are (i–p) differentiables and fuzzy numbers in $\mathbb{R}_f$, but the second derivative with respect to x is (i–p) differentiable fuzzy number inside certain limits. So, it can be concluded that the advection–dispersion equation has a fuzzy solution inside certain limits.

The profiles of reduced concentration vs. x tend asymptotically to zero, while the profiles of reduced concentration vs. t tend asymptotically to 1 ± r, where r = the spread of approximately 15%.

It is to be noted that the initial spread of r of 15% continues unchanged through the whole domain of the solution. Consequently, it is important for practical cases (solute transport in soil physics, particularly for predicting pesticides diffusion, nitrates, heavy metals, and other solutes transport), that engineers take the right decision, distinguishing the deviations of the crisp value of concentration from the fuzzy ones, which here is accepted initially 15% and remains the same in the solution domain.

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