

Mathematical analysis of sub-atmospheric vapor pipeline transmission for seawater desalination: Green and Eigen functions solutions

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ABSTRACT

Seawater desalination by sub-atmospheric vapor pipeline transfer (known as SAVP) is one of the innovative seawater desalination methods that could be used in lands and industries. SAVP systems works based on the temperature difference between a hot source and a cold environment; this method can provide users with a variety of advantages in industrial and field applications. The temperature of the hot and cold sources, as boundary conditions, can be considered as a function of time in natural and industrial environments; therefore, it affects the process of convection and diffusion significantly. In such a case, new and interesting challenges arise; such as reviewing and simplifying the basic convection–diffusion equation through obtaining the temperature profile in the pipeline by using advanced engineering mathematics. Two mathematical approaches can be developed to solve the temperature differential equation; one is through Eigen functions and the other one uses Green's equation based on the length of the pipeline for SAVP. In this study, vapor's temperature will be formulated as a function of time and length in accordance with the given assumptions and information. Mathematical simulations were performed for a field-scale spanning between Bandar Abbas and Geno (two places on the south coast of Iran) biosphere reserve. Also, an industrial scale in a vapor transfer device with a heat source capacity of about 200 L was assembled. The results showed an acceptable range of accuracy in the proposed methods.

Keywords: Sub-atmospheric vapor pipeline; Seawater desalination; Mathematical analysis; Eigen functions; Green function; Temperature difference

1. Introduction

Sub-atmospheric vapor pipeline (SAVP) and vapor condensation at destination have been of interest for seawater desalination and provision of drinking water supply after 2000 (mainly because of their newly confirmed advantages); and have been the subject of several experimental and theoretical studies till now. The use of natural potentials makes desalination and seawater transfer in arid

and water-scarce areas highly attractive and justifiable. However, the limited rate of freshwater transfer and the need for natural warm and cold sources for implementation are obvious limitations of the SAVP method; however, this method has several benefits and advantages in terms of industrial efficiency indices and economic criteria. Drastic reduction or even elimination of energy consumption, advantages such as seawater desalination with minimal energy input by eliminating the heat source and

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mechanical devices, the drastic reduction of energy consumption (or elimination) in the process of water transfer, the possibility of re-access to energy at destination through the use of a water turbine and finally utilizing the benefits of the Paris Convention for the advantage of not producing carbon dioxide make the SAVP desalination highly efficient and preferred over other methods [1,2]. In another invention called “desalination towers”, which is significantly different from other available methods, hot air, after being blown and mixed with seawater, turns it into a two-phase evaporating fluid and stimulates it to go up a tower called the desalination tower [3,4].

Desalination with multi-stage flash distillation (MSF) is done inside a narrow cylindrical chamber [5]. In this method, hot and high-pressure air passes through the interlocking parts and ducts of a cylindrical chamber. This cylindrical chamber is a relatively large evaporator whose core is filled with seawater. Blowing hot air causes seawater to evaporate and subsequently be saturated. In another method called “desalination pipes” the researchers installed pipes in which the salt water of the sea heats up and evaporates after continuous passage and exposure to sunlight [6].

The closed ejector-heating-cooling cycle is another invention, which is still one of the most common desalination methods [7,8]. Among the low-energy desalination approaches, an innovation is mentioned which consists of two ten-meter columns, one designed and installed as an evaporator and the other as a condenser [9]. A portion of freshwater passes through an ejector and then, according to Bernoulli’s law, the velocity of the fluid increases at the convergent cross-section of the ejector, and consequently the pressure decreases. The seawater, with its temperature raised by a heater, becomes ready to evaporate and passes through the ejector with fresh water. The two-phase fluid of vapor and freshwater is then cooled and condensed in a heat exchanger.

The main cause of SAVP transfer is the gradient of temperature between the source and destination. The temperature, pressure, and density of the compressible fluid (which is water vapor) decreases along the way, and the vapor that enters the cold zone slowly condenses, and freshwater is produced and accumulated. This technique can be used in a factory or a tall building (industrial scale); as well as in a field application with several kilometers of distance between a coastal city and the surrounding cool altitudes (field scale), as previously studied for some Middle Eastern cities [10]. To summarize the history and background of previous works, schematic illustrations of this desalination technique used for industrial and field applications are shown in Figs. 1 and 2, respectively. These figures help readers to understand the mechanism and elements of SAVP desalination.

Some of the facts related to this transfer method, such as the mass balance of the transferred vapor [11] or the momentum balance equation have been described in previous studies [10]. In the first study, only the vapor mass balance was investigated through a few experimental relationships, and in the latter, the momentum equation

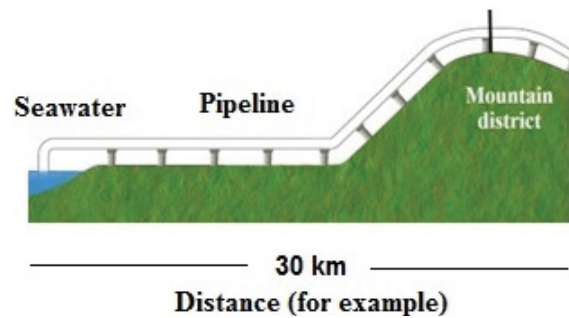


Fig. 2. SAVP technique used for a field application.

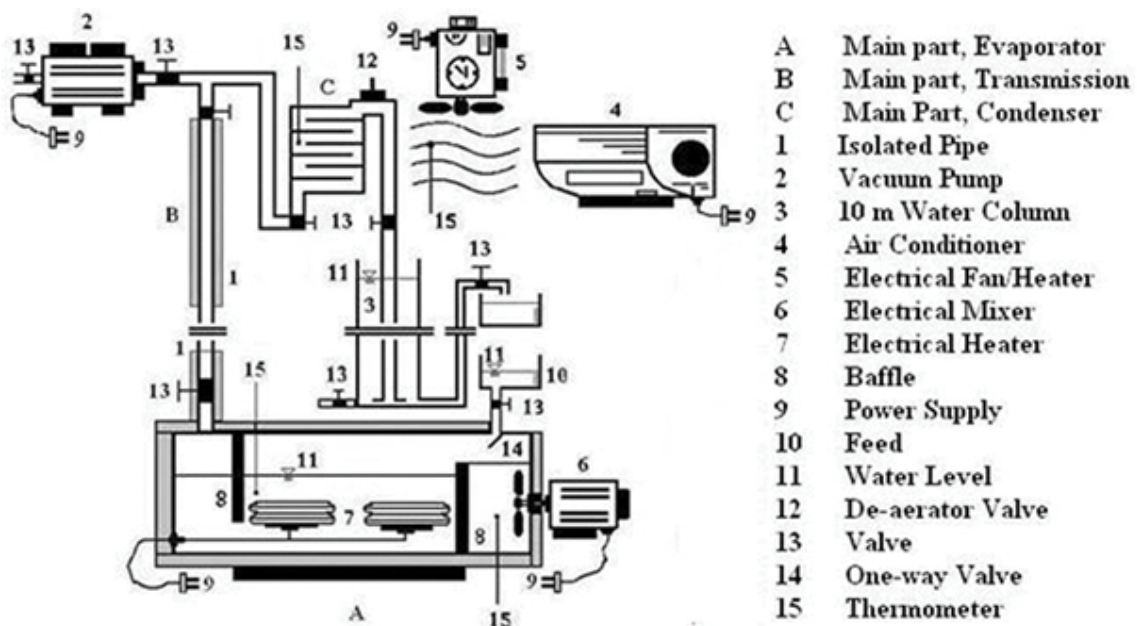


Fig. 1. SAVP technique used for an industrial application.

was solved using Antoine's equations, continuity and an appropriate equation of state, which led to the extraction of temperature and pressure profiles. However, more complete results are still expected for SAVP. In our other attempt, the fluid thermodynamics of vapor transmission through the pipeline were studied [12]. Generally, more attention needs to be paid to energy dissipations, thermodynamic losses [13] and how to minimize them in desalination [14], as well as its global commercialization [15]. Several other interrelated work based on the effect of temperature difference on sub-atmospheric vapor transmission have been done in this respect [16,17].

There are two shortcomings in the interrelated studies referenced above. First, both cases [10,11] have dealt with empirical equations in some way and still require an independent and robust mathematical framework or theory; and second, these studies have been conducted by under the assumption that the temperature is constant at the source and destination. This assumption is a result of an incomplete understanding of the nature of SAVP. Whether in a field study where the climate conditions change frequently or in an industrial scale, where the source and destination temperatures may fluctuate even more rapidly, it is no longer possible to use the equations derived from mass or momentum balances and still obtain accurate results.

The mathematical solution to such a problem involves investigating the transfer of sub-atmospheric vapor through a partial differential equation whose boundary temperatures change over a pre-known or applicable pattern. Solving such problems poses special mathematical and engineering challenges that are not easily solved through simple solutions compared to partial differential equations. The present study aims to provide an acceptable mathematical methodology for finding the temperature profile of the vapor transfer path in SAVP desalination plants on both industrial and field scales by using previous experimental and laboratory data.

2. Material and methods

2.1. Description of SAVP system

Before the mathematical modeling of SAVP for water desalination, it is necessary to have a description of the materials and equipment required for this work. An evaporator usually works with the sun's heat energy in a hot source and produces vapor. By evacuation (once only) during the life of such a desalination plant, which is done artificially, vapor rises along the pipe and condenses in the cold source. Following this, the condensation provides an automatic vacuum for vapor transference. Details of field and industrial-scale transfers have already been mentioned in the references; and temperature, pressure, and velocity profiles are available in the industrial system, which is used in this paper to verify the mathematical model [10,12]. In an industrial application, a condenser with a high level of condensation produces freshwater; while in a field application, the extension and expansion of the transfer pipe act as a tube condenser and, in a determinable range, cause the vapor to turn into liquid.

2.2. Modeling of momentum transmission phenomenon in SAVP

The vapor moves as a compressible fluid in the sub-atmospheric vapor pipe transfer and gradually dilutes during pressure and temperature drop to condensate. Due to the compressibility of the vapor, the velocity of the fluid in this transfer increases. A complete description of momentum transmission modeling is given by the study of Shojaei et al. [10]. In this modeling, the pipe length was divided into very small sections and the momentum equilibrium equation was solved along with some experimental equations such as the Antoine equation, continuity equation, and an equation of state for gases, with a trial and error technique. Fig. 3 and Eq. (1) present basics of momentum calculations. The purpose of this modeling was to determine the temperature, pressure, and velocity profiles in boundary conditions of constant and variable temperatures.

$$\frac{\partial p}{\partial t} + \frac{\partial(\rho V)}{\partial x} = 0 \quad (1)$$

2.3. Convection–diffusion equation

2.3.1. Convection–diffusion equation in SAVP

Fluid kinetic energy, some of its physical properties (such as density), friction, shear stresses, and static pressure in the form of the three known geometries of Cartesian (x,y,z), Cylindrical (r,θ,z), and Spherical (r,θ,ϕ) are in balance with each other. The ratio of length to the diameter of the pipe (L/D) is infinite. Although the tube is cylindrical, due to the insignificance of the gradient of temperature profile along the radius r and the dominancy of the temperature change in the x -direction, the Cartesian system is suitable for study and modeling.

2.3.2. Convection–diffusion equation and math operation in SAVP

Vapor transfer in a pipe can be expressed by the general convection–diffusion equation. Temperature changes as a function of time and location are expressed in this equation, in which the temperature changes due to variation in fluid velocity, its penetration, and influence of external heat or cold sources. Eq. (2) shows the convection–diffusion equation in the vapor transmission line.

$$\frac{\partial T}{\partial t} + V_x \frac{\partial T}{\partial x} + V_y \frac{\partial T}{\partial y} + V_z \frac{\partial T}{\partial z} = \beta \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \pm \frac{Q}{\rho c} \quad (2)$$

where T is the vapor temperature and t is the time, β is the coefficient of the heat fraction in the diffusion heat transfer state (in the fluid mass), ρ is the vapor density, and c is the vapor heat capacity. β is one of the physical properties of a fluid and is equal to the fraction $k/(\rho \cdot c)$ where k is the coefficient of thermal conductivity. β has a unit of area per time (e.g., m^2/s) which indicates the amount of heat diffusion. For water vapor (as a gas), the heat diffusion number is a slight function of temperature and slowly varies between 25°C and 300°C in the range of 1.22×10^{-6} to $1.32 \times 10^{-6} m^2/s$. Such small changes suggest to researchers

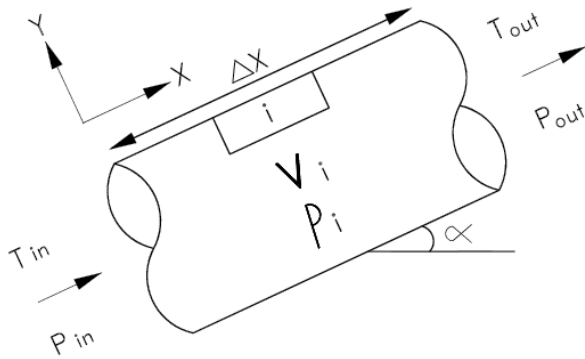


Fig. 3. An element of SAVP for mass and momentum calculations.

that this physical property, especially for cold vapor transfer, is a fixed number that will vary just based on the fluid type [18].

V is the velocity of the fluid and can be applied in all three directions of length (V_x), width (V_y), and height (V_z). Q is an external thermal factor that can affect the solution of Eq. (2) in terms of changes in time and space (or both). The algebraic sign of this thermal factor (Q) will be positive in the form of heating, and negative if it reduces the heat of the system. The most important symbol inserted in Eq. (2) is “ \pm ”, which indicates the reception or loss of heat. “System” here means the vapor transfer pipe. Eq. (2) can be simply re-written as Eq. (3), which corresponds to the SAVP and is suitable for application in mathematical solutions:

$$\frac{\partial T}{\partial t} + V_x \frac{\partial T}{\partial x} = \beta \left(\frac{\partial^2 T}{\partial x^2} \right) \pm \frac{Q}{\rho c} \quad (3)$$

If the transfer pipe is assumed to be completely insulated, the Q term will be omitted. In Eq. (3), the heat transfer of vapor whose temperature is denoted by T , occurs both in the form of heat diffusion (β) and convection (with velocity V_x). As mentioned in the introduction, the methods of Eigen functions and solving through the Green equation will be explained and utilized in the following sections. In both methods, a technique must be introduced to eliminate the V_x , solve the partial differential equation, and then insert the velocity in the final solution. The thermo-physical property of β , in Eq. (3) is not dimensionless.

2.4. Solving method using the Eigen functions

The method of solving using Eigen functions is generally a mathematical representation of the well-known method of separating variables used for non-homogeneous boundary conditions. Non-homogeneous boundary conditions usually refer to conditions where one boundary condition or both are a function of time. Due to the spatial limitation of SAVP and the specificity of its beginning and end, the boundary conditions will be consistent with the temperature conditions of the evaporating and condensing equipment. In unstable conditions, the temperature of these two pieces of equipment is a function of

time. Such a situation is consistent with the real nature of SAVP due to the fluctuations of day and night as well as the temperature fluctuations expected during different seasons. Therefore, the system of differential equations of the parts corresponding to the mentioned fact can be expressed by the system of Eq. (4). Meanwhile, it should be noted that α^2 appears for β parameter in Eq. (4).

$$\text{PDE } \frac{\partial T}{\partial t} + V_x \frac{\partial U}{\partial x} = \alpha^2 \left(\frac{\partial^2 U}{\partial x^2} \right); 0 < x < L, t > 0 \quad (4-1)$$

$$\text{BC}_1 U(0, t) = f_1(t); \quad t > 0 \quad (4-2)$$

$$\text{BC}_2 U(L, t) = f_2(t); \quad t > 0 \quad (4-3)$$

$$\text{IC } U(x, 0) = \Phi(x); \quad t = 0, 0 \leq x \leq L \quad (4-4)$$

U (the dependent function) here represents the temperature for performing and explaining mathematical operations. The above-mentioned differential equations can be considered as the most realistic and natural convection-diffusion equation to describe the temperature profile in a SAVP. For ease of performing mathematical operations, the index x is removed from V_x and the sides of the equation are divided by α^2 . Then we have:

$$\text{PDE } \frac{1}{\alpha^2} \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} - \frac{V}{\alpha^2} \frac{\partial U}{\partial x}; 0 < x < L, t > 0 \quad (5-1)$$

$$\text{BC}_1 U(0, t) = f_1(t); \quad t > 0 \quad (5-2)$$

$$\text{BC}_2 U(L, t) = f_2(t); \quad t > 0 \quad (5-3)$$

$$\text{IC } U(x, 0) = \Phi(x); \quad t = 0, 0 \leq x \leq L \quad (5-4)$$

The term $\frac{V}{\alpha^2} \frac{\partial U}{\partial x}$ makes some obstacles in the solution of Eq. (5). By removing the coefficient V/α^2 , a situation is created in which it will be possible to solve this differential equation. In other words, Eq. (5) cannot be solved analytically as long as it includes both the term $\frac{\partial U}{\partial x}$ and its coefficient V/α^2 . The technique [19] of elimination of $\frac{\partial U}{\partial x}$

will be possible through an innovative variable change as follows, which applies to the differential equation (PDE), the boundary conditions (BCs), and the initial conditions (IC).

$$\text{PDE } U(x, t) = W(x, t) \cdot \exp \left\{ \frac{V \left(x - \frac{Vt}{2} \right)}{2\alpha^2} \right\} \quad (6-1)$$

$$\text{BC}_1 U(0, t) = W(0, t) \cdot \exp \left\{ \frac{-V^2 t}{4\alpha^2} \right\} \quad (6-2)$$

$$BC_2 \quad U(L,t) = W(0,t) \cdot \exp\left\{\frac{V\left(L - \frac{Vt}{2}\right)}{2\alpha^2}\right\} \quad (6-3)$$

$$IC \quad U(x,0) = W(x,0) \cdot \exp\left\{\frac{V \cdot x}{2\alpha^2}\right\} \quad (6-4)$$

Then, a new differential equation system in terms of $W(x,t)$ with the following properties is obtained:

$$PDE \quad \frac{\partial W}{\partial t} = \alpha^2 \frac{\partial^2 W}{\partial x^2}; 0 < x < L, t > 0 \quad (7-1)$$

$$BC_1 \quad W(0,t) = f_1(t) / \exp\left\{\frac{-V^2 t}{4\alpha^2}\right\} = g_1(t); t > 0 \quad (7-2)$$

$$BC_2 \quad W(L,t) = f_2(t) / \exp\left\{\frac{V\left(L - \frac{Vt}{2}\right)}{2\alpha^2}\right\} = g_2(t); t > 0 \quad (7-3)$$

$$IC \quad W(x,0) = \frac{\Phi(x)}{\exp\left\{\frac{V \cdot x}{2\alpha^2}\right\}} = K(x); t = 0, 0 \leq x \leq L \quad (7-4)$$

Eq. (7) can generally include convective effects at the end boundary. For the evaporator, due to the limited temperature changes of the seawater as well as the beginning of vapor movement from zero velocity, convection factors usually have little effect on the vapor temperature. For the condenser, on the other hand, the velocity of the vapor at the condensate point is maximized, and therefore both the convection and diffusion factors are effective and debatable. The Peclet number (Pe) (which is the ratio of convective transfer phenomenon to diffusion transfer phenomenon) in two evaporating and condensing locations can be assumed to be small and very large numbers, respectively.

$$PDE \quad \frac{1}{\alpha^2} W_t = W_{xx}; \quad 0 < x < L, t > 0 \quad (8-1)$$

$$BC_1 \quad W(0,t) = g_1(t); \quad t > 0 \quad (8-2)$$

$$BC_2 \quad W_x(L,t) + hW(L,t) = g_2(t); t > 0 \quad (8-3)$$

$$IC \quad W(x,0) = J(x); \quad t = 0, 0 \leq x \leq L \quad (8-4)$$

When the boundary condition is a function of time, then the solution $W(x,t)$ can be considered as the sum of a transient function $W_{tr}(x,t)$ and a stable one $S(x,t)$. The terms tr and S represent transient and stable solutions, respectively. This concept is shown in Eq. (9).

$$W(x,t) = W_{tr}(x,t) + S(x,t) \quad (9)$$

As can be seen, the stable solution is denoted by the function $S(x,t)$; and the time function (t) is evident in it, which may be considered a contradiction. The explanation is that due to the limited temperature changes at the beginning and end of SAVP, in which the evaporator and condenser are located, respectively, the time function (t) imposes limited temperature conditions on the transmission line pipe during the long run. Hence, the limited range of the function $S(x,t)$ can be considered equivalent to its persistence. Such an inference has already been used in references as a solution with a solid theoretical basis [19]. A stable solution is more a function of boundary conditions than a transient solution, therefore, according to Eq. (10) we have:

$$S(x,t) = A(t) \cdot \left(1 - \frac{x}{L}\right) + B(t) \cdot \left(\frac{x}{L}\right) \quad (10)$$

The function $S(x,t)$ is of type W . Therefore, it can be re-written in a manner similar to the system of Eq. (11):

$$PDE \quad \frac{1}{\alpha^2} S_t = S_{xx}; \quad 0 < x < L, t > 0 \quad (11-1)$$

$$BC_1 \quad S(0,t) = g_1(t); \quad t > 0 \quad (11-2)$$

$$BC_2 \quad S_x(L,t) + hS(L,t) = g_2(t); t > 0 \quad (11-3)$$

The initial condition S is the same as the initial condition W . The time functions $B(t)$ and $A(t)$ and even the stable function $S(x,t)$ are extracted after the necessary placement as follows:

$$A(t) = g_1(t) \quad (12-1)$$

$$B(t) = \frac{1}{1 + Lh} [g_1(t) + Lg_2(t)] \quad (12-2)$$

$$S(x,t) = g_1(t) \left(1 - \frac{x}{L}\right) + \frac{1}{1 + Lh} [g_1(t) + Lg_2(t)] \left(\frac{x}{L}\right) \quad (12-3)$$

And subsequently, the transient part of the solution, $W_{tr}(x,t)$, will be equivalent to the following expression:

$$W_{tr}(x,t) = W(x,t) - S(x,t) \quad (13)$$

Now we have a new differential equation for the transient part, as the set of Eq. (14).

$$PDE \quad W_{tr,t} = \alpha^2 W_{tr,xx} - S_t(x,t); \quad 0 < x < L, t > 0 \quad (14-1)$$

$$BC_1 \quad W_{tr}(0,t) = g_1(t) - g_1(t) = 0; t > 0 \quad (14-2)$$

$$BC_2 \quad W_{x,tr}(L,t) + hW_{tr}(L,t) = 0; t > 0 \quad (14-3)$$

$$IC W_{tr}(x,0) = \Phi(x) - S(x,0) = \Phi_{tr}(x); t = 0, 0 \leq x \leq L \quad (14-4)$$

Eq. (14) show that the boundary conditions are homogeneous, but the non-homogeneity penetrates the PDE and appears in the form of $S_i(x,t)$ term. By clearing $\Phi(x)$ and $S(x,0)$, the expression $\Phi_{tr}(x)$ will also be known and obtained. Using Eigen functions solution, which will be explained in the following section, $W_{tr}(x,t)$ is calculated and the main functions $W(x,t)$ and $U(x,t)$ are also determined.

2.4.1. Solving the non-homogeneity for SAVP's PDE through the Eigen functions

Given the heterogeneous term $f(x,t)$ in the SAVP differential equations, the following device is considered:

$$PDE J_t = \alpha^2 J_{xx} + f(x,t); 0 < x < L, t > 0 \quad (15-1)$$

$$BC_1 J(0,t) = 0; \quad t > 0 \quad (15-2)$$

$$BC_2 J(L,t) = 0; \quad t > 0 \quad (15-3)$$

$$IC J(x,0) = \Phi(x); \quad t = 0, 0 \leq x \leq L \quad (15-4)$$

As J (in the case of $f(x,t)$) is zero, in the method of separating the variables from the product of multiplication $T(t)$. $X(x)$ is considered, given the linear presence of $f(x,t)$ in above PDE, this function can also be assumed as the product of space and time functions:

$$f(x,t) = f(t) \cdot X(x) \quad (16)$$

The spatial function is still assumed to be of the Sin type. Therefore, the heterogeneous term is expressed as Eqs. (17-1) and (17-2):

$$f(x,t) = f(t) \cdot \sin\left(\frac{n\pi}{L}x\right) \quad (17-1)$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \cdot \sin\left(\frac{n\pi}{L}x\right) \quad (17-2)$$

In the previous equation, only $f_n(t)$ is unknown; which is obtained considering the orthogonality of Sin as follows:

$$f_n(t) = \frac{2}{L} \int_{\xi=0}^{\xi=L} f(\xi,t) \cdot \sin\left(\frac{m\pi}{L}\xi\right) d\xi \quad (18)$$

Therefore the solution $J(x,t)$ (and in other words, the solution W_{tr}) can be written as follows:

$$J(x,t) = \sum_{n=1}^{\infty} T_n(t) \cdot \sin\left(\frac{n\pi}{L}x\right) \quad (19)$$

Derivatives J , that is, J_t and J_{xx} , are inserted in Eqs. (20) and (21).

$$J_t = \sum_{n=1}^{\infty} T'_n(t) \cdot \sin\left(\frac{n\pi}{L}x\right) \quad (20)$$

$$J_{xx} = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 \cdot T_n(t) \cdot \sin\left(\frac{n\pi}{L}x\right) \quad (21)$$

By placing $J_t = \alpha^2 J_{xx} + f(x,t)$ in the equation, we have:

$$\sum_{n=1}^{\infty} T'_n(t) \cdot \sin\left(\frac{n\pi}{L}x\right) = -\alpha^2 \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 T_n(t) \cdot \sin\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} f_n(t) \cdot \sin\left(\frac{n\pi}{L}x\right) \quad (22)$$

And Eq. (23) is attained.

$$\sum_{n=1}^{\infty} \left[T'_n(t) + \left(\frac{n\pi}{L}\alpha\right)^2 \cdot T_n(t) - f_n(t) \right] \cdot \sin\left(\frac{n\pi}{L}x\right) = 0 \quad (23)$$

Given the linear independence of Sin whose arcs are such that all Sins cannot become zero simultaneously:

$$T'_n(t) + \left(\frac{n\pi}{L}\alpha\right)^2 \cdot T_n(t) - f_n(t) = 0 \quad (24)$$

In this way n ODEs appear that need to be solved with n initial conditions. The n initial conditions are obtained as follows:

$$J(x,t) = \sum_{n=1}^{\infty} T_n(t) \cdot \sin\left(\frac{n\pi}{L}x\right) \quad (25-1)$$

$$\Phi(x) = J(x,0) = \sum_{n=1}^{\infty} T_n(0) \cdot \sin\left(\frac{n\pi}{L}x\right) \quad (25-2)$$

$$\int_{x=0}^{x=L} \Phi(x) \cdot \sin\left(\frac{m\pi}{L}x\right) dx = \int_{x=0}^{x=L} T_n(0) \sin\left(\frac{n\pi}{L}x\right) \cdot \sin\left(\frac{m\pi}{L}x\right) dx \quad (25-3)$$

where we have:

$$T_n(0) = \frac{2}{L} \int_{\xi=0}^{\xi=L} \Phi(\xi) \cdot \sin\left(\frac{m\pi}{L}\xi\right) d\xi; n = 0, 1, 2, \dots \quad (26)$$

At this stage the following ODE system should be solved:

$$T'_n(t) + \left(\frac{n\pi}{L}\alpha\right)^2 \cdot T_n(t) - f_n(t) = 0 \quad (27-1)$$

$$T_n(0) = \frac{2}{L} \int_{\xi=0}^{\xi=L} \Phi(\xi) \cdot \sin\left(\frac{m\pi}{L}\xi\right) d\xi; n = 0, 1, 2, \dots \quad (27-2)$$

The solution of these ODE systems will be available through the following pattern:

$$\text{ODE } y'(t) + a(t) \cdot y(t) = b(t); t > 0 \tag{28-1}$$

$$\text{IC } y(0) = \text{Known}; t = 0 \tag{28-2}$$

Therefore:

$$y(t) = e^{-a(t)t} \cdot y(0) + \int_{\tau=0}^{\tau=t} b(\tau) \cdot e^{-a(t)(t-\tau)} d\tau \tag{29}$$

The ODEs solution of $T_n(t)$ is equal to:

$$T_n(t) = T_n(0) \cdot e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} + \int_0^t f_n(\tau) \cdot e^{-\left(\frac{n\pi\alpha}{L}\right)^2 (t-\tau)} d\tau \tag{30}$$

After placement, we have:

$$T_n(t) = \frac{2}{L} \left(\int_0^L \text{Sin}\left(\frac{n\pi}{L}\xi\right) \cdot \Phi(\xi) d(\xi) e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \right) + \frac{2}{L} \left(\int_0^t e^{-\left(\frac{n\pi\alpha}{L}\right)^2 (t-\tau)} \cdot \int_0^L f(\xi, \tau) \cdot \text{Sin}\left(\frac{n\pi}{L}\xi\right) d\xi d\tau \right) \tag{31}$$

The final solution is presented as the following equation:

$$J(x, t) = \sum_{n=1}^{\infty} T_n(t) \cdot \text{Sin}\left(\frac{n\pi}{L}x\right) \tag{32}$$

2.5. Solving the convection–diffusion differential equation in SAVP through Green’s function

If the temperature profile changes in SAVP are to be calculated and followed dynamically by considering instantaneous changes in both the boundary conditions and the initial condition, the convection–diffusion differential equation will have to be solved through the Green function. Unlike the solution of the Eigen functions in which homogeneous PDEs with heterogeneous boundary conditions and the time function lead to a level of interference response in an equation involving double integrals with the virtual variables time (τ) and location (ξ), the Green function can solve a heterogeneous PDE whose heterogeneity term is a function of time and location ($F(x, t)$) and its boundary conditions are a function of time, by separating the effect of each variable in the final answer.

But what does the term $F(x, t)$ in PDE mean, when the terms of their boundary conditions are time-dependent and heterogeneous? $F(x, t)$ can indicate any instability that upsets the SAVP system. These instabilities, such as incomplete insulation and SAVP going out of equilibrium and moving to a new equilibrium state, are examples of operation in a transient state. In other words, the differential equation of heat transfers through the Green function, with a special application, examines SAVPs during transition from one equilibrium state to another.

This application focuses on the transition state of saturated vapor from one thermodynamic equilibrium to another, which inevitably introduces non-equilibrium thermodynamics.

Although the convection–diffusion equation with maximum unstable (transient) terms is studied in this section, the subject of non-equilibrium thermodynamics cannot be included comprehensively in this study. One of the prominent features of non-equilibrium thermodynamics is entropy production due to the constant moving away and approaching equilibrium, which is well expressed and formulated for various systems, including systems in which heat transfer is accompanied by entropy production [20].

As mentioned earlier, all spatial and temporal terms derived from the non-equilibrium or transient approach can be accumulated in $F(x, t)$. It is noteworthy that the fluid in the energy dissipation equation is introduced as a compressible fluid. In other words, the assumption of incompressible fluid prevents the user from entering the realm of non-equilibrium thermodynamic calculations.

Eq. (33) shows a complete SAVP whose velocity has already been eliminated by the aforementioned techniques. This equation is introduced as the basis of PDE and its solution through the Green function. A few points should be noted before going through the solution method via the Green function. First: the velocity term is omitted and assumed to be removed from boundary conditions, in a similar manner as in the technique introduced earlier. Therefore, this PDE is not a PDE without fluid velocity. Second: the correlation of temperature profile solution in SAVP with the solution method through Green function is strongly dependent on fluid compressibility. Third: T , instead of U , is considered as the principal variable in solving through the Green function.

$$\text{PDE } \frac{\partial T}{\partial t} = \alpha^2 \frac{\partial^2 T}{\partial x^2} + F(x, t); \quad 0 < x < L, t > 0 \tag{33-1}$$

$$\text{BC}_1 T(0, t) = T_{\text{Seawater}} = f_1(t); \quad t > 0 \tag{33-2}$$

$$\text{BC}_2 T(L, t) = T_{\text{Hill}} = f_2(t); \quad t > 0 \tag{33-3}$$

$$\text{IC } T(x, 0) = \Phi(x); \quad t = 0, 0 \leq x \leq L \tag{33-4}$$

The solution of this equation will be as Eq. (34), in which the Green function (G) is seen in all terms.

$$T(x, t) = \int_0^L G(x, t | \xi, \tau) \Big|_{\tau=0} \cdot \Phi(\xi) d\xi + \text{Effect of Initial Condition} \tag{34-1}$$

$$\int_{\tau=0}^{\tau=t} \int_{\xi=0}^{\xi=L} G(x, t | \xi, \tau) \cdot F(\xi, \tau) d\xi \cdot d\tau + \text{Effect of the Non-Homogenous Term} \tag{34-2}$$

$$\int_0^t \frac{\partial G}{\partial \xi} \Big|_{x=0} \cdot f_1(\tau) d\tau - \text{Effect of First Boundary Condition} \tag{34-3}$$

$$\int_0^t \frac{\partial G}{\partial \xi} \Big|_{x=L} \cdot f_2(\tau) d\tau - \text{Effect of Second Boundary Condition} \tag{34-4}$$

The first sentence in Eq. (34) expresses the effect of IC; the second expresses the effect of heterogeneity $F(x,t)$; the third expresses the influence of BC1 and the fourth expresses the effect of BC2. Unlike the solving methodology using Eigen functions, which is not able to distinguish the effects of different terms in the final answer, the Green function, as stated in Eq. (34), can accurately separate the effects of temporal and location terms that have appeared heterogeneously through analytical separation.

2.5.1. Determining the Green's function, $G(x,t|\xi,\tau)$

Green function G is the function of location (x), time (t), and some dummy variables, that is, ξ and τ . The appearance of partial derivatives of the Green function at the boundaries is aimed to express and consider all potential changes in the Green function at that boundary over time. On the other hand, the appearance of a negative algebraic sign (-) in the last sentence indicates that the definition of the derivative in the Green function is forward, which is defined from the beginning in the basic concepts of the Green function and is shown in Eq. (35).

$$G(x,t|\xi,\tau) = \frac{2}{L} \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi\alpha}{L}\right)^2(t-\tau)} \cdot \sin\left(\frac{n\pi\alpha}{L} \cdot x\right) \cdot \sin\left(\frac{n\pi\alpha}{L} \cdot \xi\right) \quad (35)$$

When the Green function is specified, the solution $T(x,t)$ is specified dynamically.

3. Results and discussion

3.1. Calculation and discussion of SAVP through the energy equation for the laboratory system

In the SAVP laboratory system, mild or severe time functions can be adopted for boundary conditions. Mild

and severe temperature changes indicate whether the whole SAVP will reach thermodynamic equilibrium or is it still in the unstable dynamic state of continuous temperature changes along the pipe after each change in the source and destination temperatures. These two modes can be practically studied due to the limited length of the pipe.

Mathematically, the Eigen functions can simply predict the true temperature profile for slight temperature changes at the boundaries. Therefore, the steady-state thermodynamics assumption is consistent with this situation, and Eq. (32) can be used to predict the temperature profile. Severe temperature changes also lead to a transient state in which the fluid in the tube never reaches equilibrium and is constantly moving around it. Focusing on the latter fact, Eq. (34) can be used to profile the transient temperature.

3.2. Calculation and discussion of SAVP through the convection–diffusion equation (Eigen functions)

The convection–diffusion equation for the state where SAVP has reached equilibrium was applied using the Eigen functions for both minimum and maximum temperature differences for the saltiest water ($S = 4\%$) and compared with the results of using the momentum equation. Figs. 4 and 5 show the results of solving the temperature profile using the Eigen functions compared to momentum equation. Fig. 4, which illustrates temperature changes over 48 h, demonstrated from operational experience while with each change, there was ample opportunity for SAVP to reach equilibrium. The result of applying these temperature changes is reported in Fig. 4. Fig. 5 compares the results of the temperature profile by solving the convection–diffusion equation with the solution obtained from the momentum equation. The points shown in Fig. 5 are experimental. There is an acceptable agreement between the two theoretical curves and the experimental points.

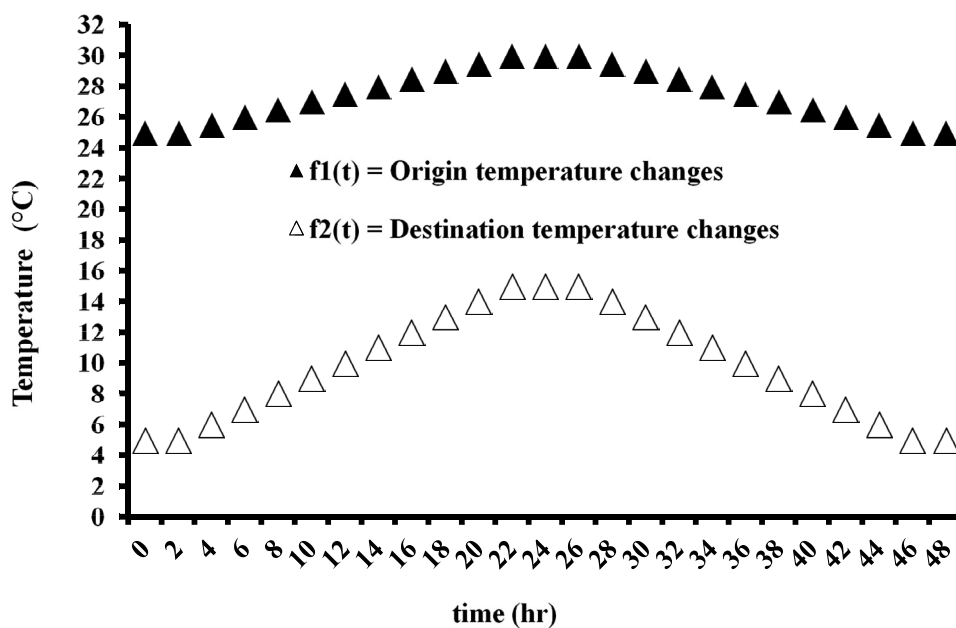


Fig. 4. Applying the temperature functions for boundaries for solving the convection–diffusion equation through the Eigen functions.

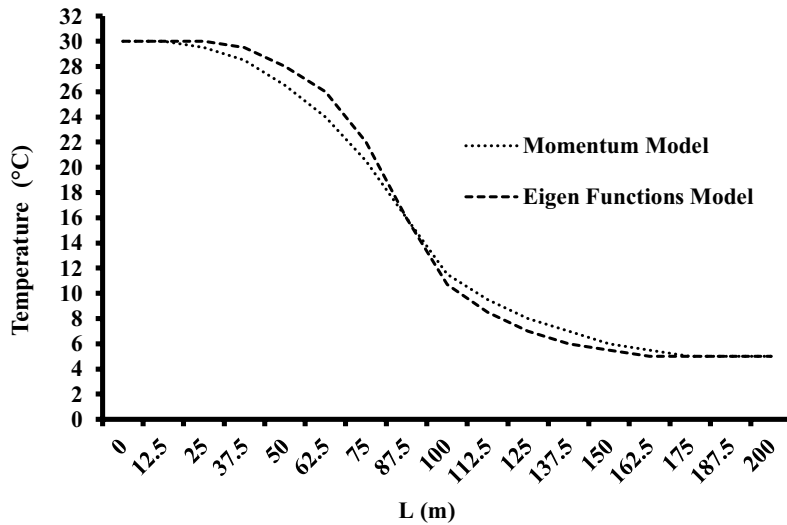


Fig. 5. Showing the temperature profile in experimental system through momentum model and solving the convection–diffusion equation through the Eigen functions for 1-inch diameter and salinity of 4.0 wt.%. The origin and destination temperature was 30°C and 5°C, respectively.

The two functions $f_1(t)$ and $f_2(t)$ are temperature functions that change with time (t) at the boundaries. These two functions can be selected as completely arbitrary functions imposed by the user on the mathematical model. Here, two parabolic shape functions that have a maximum in the middle time interval are introduced as a practical example. The duration of the application of these two time functions is such that (according to the experience attained from the operation of device) they can make the system reach equilibrium after each change that lasts 2 h. In other words, the 2 h temperature change steps are adjusted so that the laboratory desalination system can leave the steady-state equilibrium of a thermodynamic state and reach and stabilize at the new thermodynamic equilibrium. The origin of this experimental technique and its related time steps is attributed to the fact that the device used takes about an hour to reach a steady state. Therefore,

each time step for 2 h certainly provides ample opportunity for the desalination plant to reach equilibrium.

Fig. 5 is based on the equilibrium of temperature applications at the boundaries that contains temperature changes of $f_1(t)$ and $f_2(t)$, for a diameter of one inch and a salinity of four percent. The temperature profile is solved by two equations: convection–diffusion (using the Eigen functions method) and using momentum model. This figure can contain several monographs, but the present figure shows the case when $f_1(t)$ reaches 30 and $f_2(t)$ reaches 5°C.

3.3. Calculation and discussion of SAVP through the convection–diffusion equation (Green’s function)

In this case, extremely drastic changes can be made to make the SAVP conditions completely dynamic and unstable while there are no restrictions. But to understand the

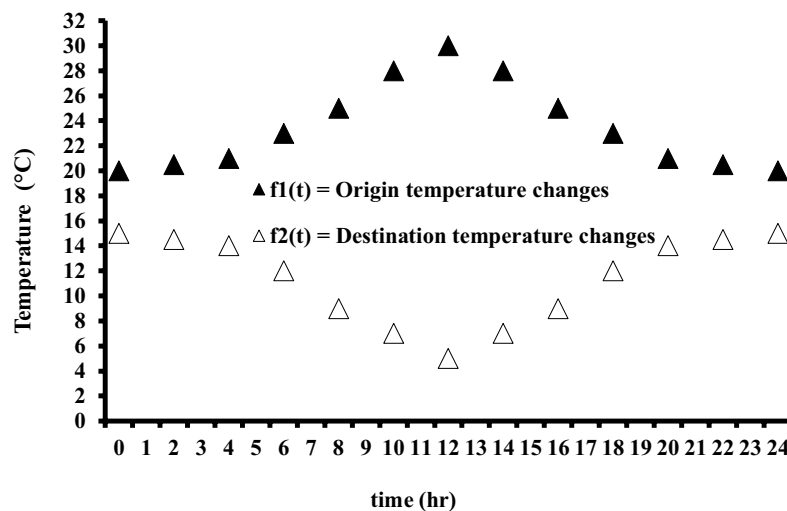


Fig. 6. Applying the temperature functions for boundaries for solving the convection–diffusion equation through the Green’s function.

problem better, Fig. 6 is suggested as a representative of the most intense temperature action at the boundaries. It is imposed on the SAVP boundaries over 24 h, a period that is shorter than what has been applied in Section 3.2 (48 h). These temperature changes at the boundaries are selected in such a way that the maximum and minimum temperatures time coincide. The two-dimensional Fig. 7 depicts the temperature profiles obtained by solving Green equation; that is, Eq. (34). Unlike Fig. 5, where the solution of the Eigen functions is compared with the solution of the momentum equation, the output of the Green function cannot be compared with the profile obtained from the momentum equation. The reason for this is that there is an assumption of equilibrium in momentum solution that contradicts the nature of Green's unstable solution. Fig. 7 shows the significant and intertwined profiles of temperature.

The lines shown in Fig. 7 are in fact isotherms of time that were first applied at the boundaries, but the consequent effects of instability are observed along the pipe. To avoid overlapping isotherms that are drawn 1 h apart, isotherms are selected and plotted that are 4 h apart. These lines show the time and location at which the temperature of the fluid varies (at the length of the pipe). Some of the experimental values, represented as black dots, correspond to an acceptable approximation on the related lines extracted from the mathematical model.

3.4. Calculation and discussion of SAVP through the convection–diffusion equation (Eigen function) for inland scale

For presentation of temperature profiles in inland scale, the pipeline reaches thermodynamic equilibrium by temperature change in boundaries (sever or mild). This, however, could be more reliable knowing the fact that in this case the temperature alteration is chiefly due

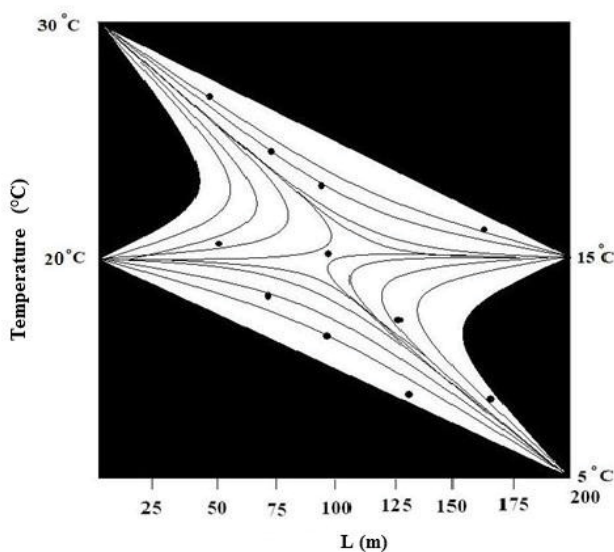


Fig. 7. Showing the temperature profile in calculated experimental system through solving the convection–diffusion equation through the Green's function for 1-inch diameter and salinity is 4.0 wt.%.

to real climatical alternations in which no sever and fast changes occur. In other words, two factors of pipeline length and smooth temperature changes cause the solution to be achieved only by the Eigen functions. In this respect, the temperature profile due to season change for the inland case study of Bandar Abbas-Geno is first presented in Fig. 8 and its solution is subsequently obtained by Eq. (32). The salinity is assumed to be 4.0 wt.% for production of Fig. 8. This profile resembles the temperature profile of the pipeline presented in our previous work based on momentum studies (Bandar Abbas-Geno) [12].

4. Conclusions

The results of this study showed that the SAVP's temperature profile can be mathematically modeled by the solution of the Eigen/Green's function for the convection–diffusion equation; and the outputs are in good agreement with the temperature profiles resulting from solution of momentum equation. In fact, for SAVP studies, the temperature profile is the key output that could link the results between the mathematical modeling and the momentum equation.

The transfer-diffusion equation was solved for the laboratory and inland system using both the Eigen functions and the Green function. These solutions are provided to show how SAVP works from two different perspectives that can confirm the accuracy of pressure drop, temperature drop, density, and fluid velocity in the pipe. In the laboratory mode, with the help of data quantifying temperature drop and velocity at the outlet, the temperature profile calculations were verified by solving the convection–diffusion equation. The convection–diffusion equation in the laboratory system was solved for mild and severe temperature changes using the Eigen functions and the Green function, respectively; and convincing results were obtained. Based on these agreements between experiments and theoretical framework, the temperature profiles study for inland scale was reviewed.

In a material-field based comparison, it is easy to observe that the mass of the materials used at the warm source and the cold destination is very small compared to the mass used for the vapor transfer line. This fact underscores the importance of investing in piping, from an engineering point of view; and basics of project implementation management should be highlighted at this stage. Despite the mentioned fact on the length of the vapor transfer pipe, the industry currently produces high-quality and lightweight pipes in different diameters with acceptable strength coefficient, mass and heat insulation, and the required expertise in installation and operation of the pipeline is in the assurance stage. Therefore, considering the return of investment at the exact long term prospective, there are no challenging obstacles or restrictions on the length of piping. By performing conventional environmental and meteorological studies, as well as mastering the temperature status of the source and destination and selecting the correct points for transfer, the vapor transfer plan through SAVP can be applied and implemented in accordance with engineering and implementation standards.

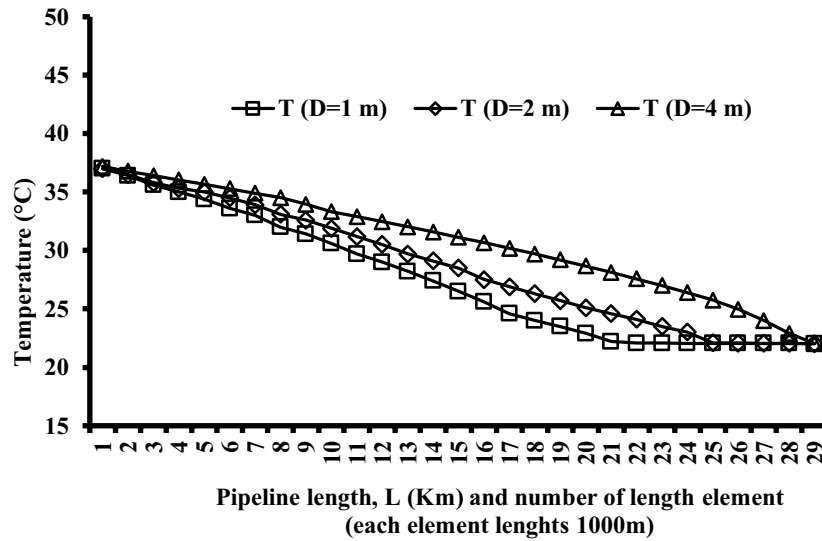


Fig. 8. Elemental partition of the vapor transfer between Bandar Abbas and Geno includes vapor temperature in each element for pipes (diameters of 1, 2, and 4 m) along the path and salinity of 4.0 wt.%.

It's worth mentioning that if the temperature in the cold source reaches zero degrees Celsius or less, the freshwater production system will be closed and disrupted due to the freezing of the condensate line. This restriction must be considered in choosing the piping site based on knowledge of climate conditions and the meteorological background of the site.

Symbols

- $a(t)$ — Time function for an ODE solution, –
- BC — Boundary condition, –
- C — Thermal heat capacity of vapour, J/(kg °C)
- $F(x,t)$ — Non-homogeneity function, –
- $f(t)$ — Time function, –
- $G(x,t|\xi,\tau)$ — Green function solution, –
- $g(t)$ — Time function, –
- H — A representative for convective heat transfer coefficient, DL
- $J(x)$ — Transformed initial condition, –
- $J(x,t)$ — Non-homogeneity solution, –
- $K(x)$ — Transformed initial condition, –
- L — Pipeline length, m or DL
- M — Numbers 0, 1, 2, ..., –
- N — Numbers 0, 1, 2, ..., –
- PDE — Partial differential equation, –
- Q — External thermal source or sink, $\pm J$
- R — Spherical or cylindrical distance variable, –
- S — Steady state solution, –
- T — Vapor temperature, °C, K or DL
- $T(t)$ — Time function, –
- T — Time, sec, h
- Tr — Transition state, subscripts
- U — Dependent function, that is, DL temperature, –
- V — Vapor velocity, m/s
- W — Homogeneous dependent function, –

- W_{tr} — Transition solution, –
- x, y, z — Length variable or vapor transfer pipeline, m
- $y(x)$ — Solution for an ODE, –

Greek letters

- B — Thermal diffusion coefficient, m^2/s
 - Θ — Spherical or cylindrical angle variable, DL
 - $\Phi(x)$ — Initial condition; a function of x , –
 - Φ — Spherical angle variable, DL
 - Ξ — Dummy variable for x , DL
 - τ — Dummy variable for t , DL
 - ρ — Vapor density, kg/m^3
 - ∂ — Partial differential symbol, DL
- *DL = dimensionless

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