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Probabilistic solution of steady-state soil water storage and plant water stress equation using cumulant expansion theory

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ABSTRACT

The advanced stochastic model of soil water and plant water stress is derived from this study by better understanding of their dynamic under climate for forcing and their role in water-controlled ecosystem than previous study by Kim et al. The distinguishing remark of this study is the loss function of dividing into two phases. The cumulant expansion theory is applied for obtaining an ensemble-average equation. The proposed model is simple yet realistic that it can account for essential features of the system. The probability distribution functions of soil water and plant water stress derived in this study are also examined to investigate how the probability distribution functions for soil water and plant water stress behave when each parameter is changed as well as mean and variance of rainfall are changed.

Keywords: Ecohydrology; Modeling; Soil; Water

1. Introduction

The interests in the dynamic relationship of the climate-soil-vegetation system have been amplified among many researchers, and especially the studies about soil water and its impact on plant water stress are vigorously in process [1–3]. The soil water has an essential role in water-controlled ecosystem, that is, its substantial influence on the structure and organization of vegetation, and the plant water stress is also a key factor that largely affects the plant's water potential under the circumstance of limiting water [4]. Therefore, the spatial and temporal understanding of soil water and plant water stress lies at the center of ecohydrology.

Many researchers have investigated deterministic hydrologic models in order to understand the soil water and plant water stress, but, de facto, the models possess limitations to include the fluctuating characteristic of precipitation. Because the observed precipitation data are generally analyzed to a stochastic manner and parameters and external forces in hydrologic models are randomly reacted, using stochastic models gives a better solution in terms of understanding such phenomena.

This study suggests the advanced stochastic models based on the study of soil water dynamics [5]. One of the distinctive points of this study is the derivation of soil water probability distribution function (PDF) with two different phases of loss term from the previous study, from which the probabilistic behavior of water stress in the aspect of vegetation can be investigated properly. In the aspect of modeling, relatively simple models can be applied to embody the essential features of soil water dynamics and the corresponding vegetation response, and particularly, the analytical solution derived from such models may give clearer understanding of the influences of important parameters. Moreover, such an analytical solution can be a cornerstone in applying to more complicate phenomena.

The soil water dynamics herein is analyzed in terms of daily timescale, and the soil layer is conceptualized by the linear reservoir of which water is intermittently provided

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by stochastically occurring rainfalls. In this study the characteristics of meteorological and soil conditions that change sub-daily are excluded. For the simpler construction of the model, the soil is assumed to be spatially homogeneous, and in daily basis, the wetting front movement in the vegetated soil when precipitation occurs and the vertical re-distribution of soil water when a rainfall event terminates are ignored. Also the only source of soil water from the soil storage is assumed to be the vertical infiltration by rainfall, and thus, the soil water at the topsoil that includes the plant's root zone is mainly considered. The evapotranspiration related with the soil water loss is assumed to be increased by the maximum evapotranspiration rate when soil water is abundant enough and linearly decreased when soil water lacks.

2. Materials and methods

2.1. Study area

The proposed model of soil water and plant water stress is applied to the city of Daegu, a basin city locacted in the southeastern Korea and known as the driest area of Korea. The city is chosen for our model application because one of the severest drought events had occurred under the Korean history in 1994 and 1995, and this period is selected for excluding the seasonality of the meteorological condition.

2.2. The stochastic model derivation of the soil water dynamic

It is assumed that the soil is spatially homogeneous, and thus, the spatial variability of the soil water dynamics for the simplification of the model is ignored; therefore, the governing equation of soil water dynamics can be simply represented as such (Yoo et al.) [14]:

$$nZ_r \frac{ds}{dt} = -L(s) + R \tag{1}$$

where n is the soil porosity; Z_r is the depth (L) of the topsoil layer including the plant's root zone; L is the loss function which depends on the conditions of climate, soil, and vegetation (L/T); R is the precipitation rate (L/T); and s is the soil water which is the state variable.

It can be more easily expressed as follows:

$$\frac{ds}{dt} = -W(s) + ZR \tag{2}$$

where,

$$W(s) = \frac{L(s)}{nZ_r} \; and \; Z = \frac{1}{nZ_r} \tag{3}$$

The most distinguishing remark of this study over previous work [5] is that the soil water loss function, $L(s)$, can be divided into two phases based on the soil water condition. That is:

$$\begin{aligned} L(s) &= \frac{E_{\max}}{s^*} s, && \text{for } 0 \le s \le s^* \\ &= E_{\max}, && \text{for } s^* \le s \le 1 \end{aligned} \tag{4}$$

where s^* is the critical soil water value below which plants start reducing transpiration by closing their stomata, and E_{max} is the potential evapotranspiration rate [L/T]. Such a two-phase soil water loss function is necessary to model the probabilistic behavior of water stress in the aspect of vegetation. In Eq. (4), the evapotranspiration rate continues at potential rate until soil water falls below s^*. Then for $s < s^*$, the evapotranspiration rate decays linearly in response to soil water in deficit of S^*.

Using cumulant expansion theory [5–10], the Fokker–Planck equation, which explains the temporal behavior of soil water PDF, can be derived as follows:

$$\begin{aligned} \frac{\partial p(s,t)}{\partial s} &= -\frac{\partial}{\partial s}\left[\left\{-W(s) + Z < R >\right\} p(s,t)\right] \\ &\quad + \frac{\partial}{\partial s}\left[\left\{\frac{1}{2} Z^2 \theta Var[R]\right\} \frac{\partial p(s,t)}{\partial s}\right] \end{aligned} \tag{5}$$

where $p(s,t)$ stands for the PDF of the state variable s at time t; $<R>$ is for the average of daily precipitation rate; $Var[R]$ is for the variance of precipitation; and θ is the scale of fluctuation of precipitation defined as follows [11]:

$$\theta = 2\int_0^{\infty} \rho(\tau) d\tau \tag{6}$$

where ρ(τ) is the autocorrelation coefficient at lag-τ days.

In order to obtain the probabilistic solution for steady-state soil water condition, the left-hand side of Eq. (5) needs to be set to zero, and that is:

$$\frac{d}{ds}\left[\left\{-W(s) + Z < R >\right\} p(s)\right] = \frac{d}{ds}\left[\left\{\frac{1}{2} Z^2 \theta \operatorname{var}\left[R\right]\right\} \frac{dp(s)}{ds}\right] \tag{7}$$

Integrating with respect to s and applying a Robin boundary condition as $(-W + Z <R>)\, p(1) = Z^2\theta \mathrm{var}[R]p'(1)/2$), one can obtain the first-order linear ordinary differential equation. Therefore, the steady-state PDF of soil water can be obtained as follows:

$$p(s) = \frac{1}{C} \exp\left[\int_0^s \left(\frac{-2W(s) + 2Z < R >}{Z\theta^2 Var[R]}\right) ds\right] \tag{8}$$

where C is the normalized constant that satisfies the following equation:

$$\int_0^1 p(s) ds = 1 \tag{9}$$

If the loss function is directly considered to the Eq. (8), the final steady-state PDF of soil water can be obtained as follows:

$$\begin{aligned} p(s) &= \frac{1}{C} \exp\left[\frac{2}{Z^2 \theta Var[R]}\left(-\frac{E_{\max} Z s^2}{2s^*} + Z < R > s\right)\right], \text{for } 0 \le s \le s^* \\ &= \frac{1}{C} \exp\left[\frac{E_{\max} Z s^* - 2sZ(E_{\max} - < R >)}{Z^2 \theta Var[R]}\right], \qquad \text{for } s^* \le s \le 1 \end{aligned} \tag{10}$$

2.3. The derivation of the steady-state PDF on the plant water stress

When the amount of water in the soil becomes so low that it can lead a physiological defect, the vegetation can suffer from water stress. That is, water stress on vegetation is principally controlled by the soil water dynamics which have stochastic features, and it can be characterized by applying to a threshold value of soil water. The plant water stress ξ can be modeled as having two different phases [12]:

$$\xi = \frac{s^* - s}{s^*}, \text{ for } 0 \le s \le s^* \quad = 0, \quad \text{for } s^* \le s \le 1 \tag{11}$$

where s^* is soil water level corresponding to incipient stomata closure. When soil water is more than the threshold value s^*, the vegetation is assumed not to suffer water stress.

Given the PDF of soil water (Eq. (10)) and the relation between soil water and plant water stress (Eq. (11)), the cumulative distribution function (CDF) of plant water stress can be obtained using derived probability distribution theory. From the CDF of plant water stress, the PDF of plant water stress can be easily obtained by differentiation.

Fig. 1 schematically shows the transformation of the PDF of soil water to the PDF of plant water stress according to the soil water-plant water stress transformation function as depicted in Eq. (11). Under the assumption of plant water stress, the vegetation will not suffer water stress if soil water is more than the threshold value s^*. As a result, the probability of vegetation with no water stress is to have the impulse probability which is equal to the probability that a given soil water exceeds the threshold value s^* (represented by the shaded area in Fig. 1). This impulse probability is given by:

$$P(\xi = 0) = P(s^* < s < 1) = -\frac{1}{C}\left[\frac{Z\theta Var[R]\left\{\exp\left(\frac{2<R>+2E_{\max}s^*}{Z\theta Var[R]}\right) - \exp\left(\frac{2<R>s^* + 2E_{\max}}{Z\theta Var[R]}\right)\right\}}{2\left(E_{\max} - <R>\right)}\right] \cdot \exp\left(-\frac{E_{\max}(2+s^*)}{Z\theta Var[R]}\right) \tag{12}$$

The remainder of plant water stress CDF exists over the range where plant water stress occurs (ξ > 0), which corresponds to the range where soil water is less than the threshold value ($s < s^*$). That is:

$$F(\xi) = \int_{s^* - s^*\varsigma}^{s^*} p(s)ds + \int_{s^*}^{1} p(s)ds \tag{13}$$

The PDF of plant water stress may be obtained as the derivative of Eq. (13) as follows:

$$f(\xi) = \frac{1}{C}\left(s^* \exp\left[-\frac{s^*(\xi-1)(E_{\max}\xi + 2<R> - E_{\max}}{Z\theta Var[R]}\right]\right) \tag{14}$$

3. Results and discussion

3.1. Application of rainfall-runoff

The main purpose of this study was to investigate the probabilistic behavior of soil water and plant water stress, and hence, the daily precipitation data from April to October known as the growing season of the vegetation were used to simulate soil water and plant water stress. Fig. 2 shows the simulation result of soil water and plant water stress.

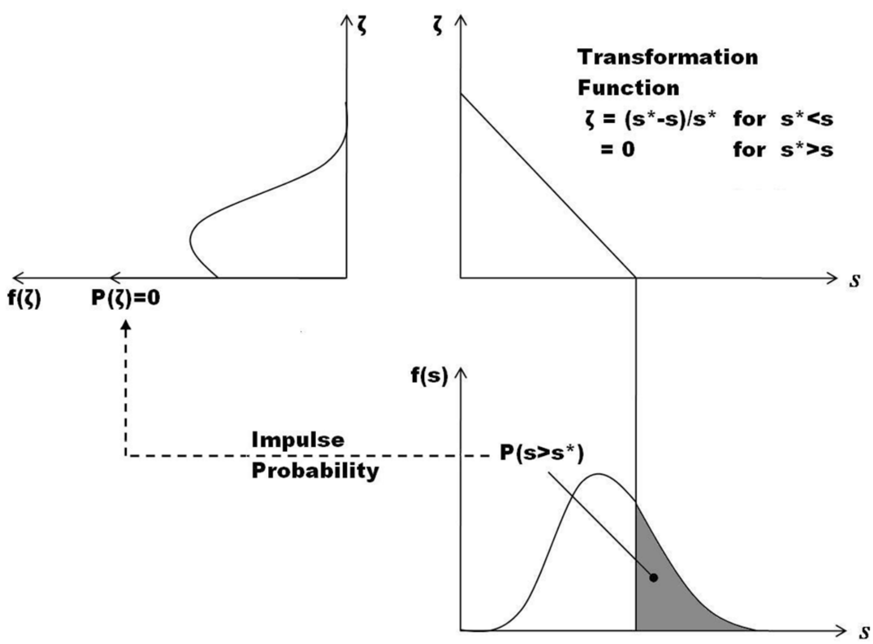

Fig. 1. Derived probability distribution function.

The figure confirms the fact that the vegetation suffererd from the lack of water when the rainfall was not abundant enough.

Each parameter needed for the simulation of soil water and plant water stress was estimated as follows. The potential evapotranspiration rate E_{max} was calculated as the averaged value from April to October in 1993 to 1995 in Daegu obtained from the Korea Meteorological Administration, and the scale of fluctuation of rainfall θ was estimated by using the Poisson Rectangular Pulse rainfall model [5]. The effective soil depth nZ_r was assumed to be 150 mm. In fact, the determination of nZ_r is rather subjective. Previous study had assumed the depth of the topsoil layer Z_r to be 500 mm and the porosity n to be 0.3 (so the nZ_r becomes 150 mm) [13]. The values adopted in this study were based on the similar assumption. The value of 150 mm for nZ_r was also used in [5, 14]. The selected parameter values were as follows:

$$\langle R \rangle = 2.5811 \text{ mm/day}$$
$$Var[R] = 55.5759 \text{ (mm/day)}^2$$
$$\theta = 0.9254 \text{ day}$$
$$E_{max} = 4.6164 \text{ mm/day}$$
$$nZ_r = 150 \text{ mm}$$

Also, the threshold value of plant water stress was chosen as 0.5 for further simulation. Here, when equals to 1, the result should become exactly as same as the 1 [5].

The PDF was calculated after simulating soil water time series, and the results are shown in Fig. 3. The black histogram in Fig. 3 represents the relative frequency histogram based on the numerically simulated soil water. The observed daily precipitation data used for modeling soil water is three-year data from 1993 to 1995, and the 1993 data were used as warming-up. Therefore, only 1994 and 1995 data were actually analyzed. As shown in Fig. 3, the analytically derived steady-state soil water PDF (Eq. (10)) has a good agreement to the numerically generated soil water data.

Fig. 4 shows the relative frequency histogram of plant water stress generated numerically, and the steady-state plant water stress PDF derived by Eq. (14). From this figure, the analytically derived steady-state plant water stress PDF was in a good accordance with numerically generated probability distribution. In addition, it is noticeable that the PDF of plant water stress has probability mass at ξ = 0 as mentioned before and that such probability mass is shown to be reproduced well.

In order to better understand the response of soil water system to its parameters' variability, a simple sensitivity analysis was carried out. Fig. 5 shows the PDF of soil water for three different values of mean rainfall $E[R]$ while holding other parameters unchanged. As can be seen in Fig. 5, if the average of daily rainfall would be increased and less than daily potential evapotranspiration rate, the overall shape of steady-state soil water PDF would not be changed. However, if the average of daily rainfall would be increased and larger than daily potential evapotranspiration rate, the overall shape of steady-state soil water PDF

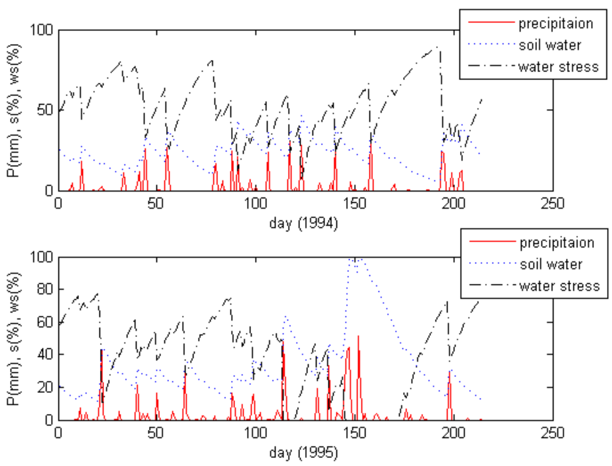

Fig. 2. Simulation of soil water and water stress using observed daily rainfall data in 1994 and 1995 in Daegu, Korea.

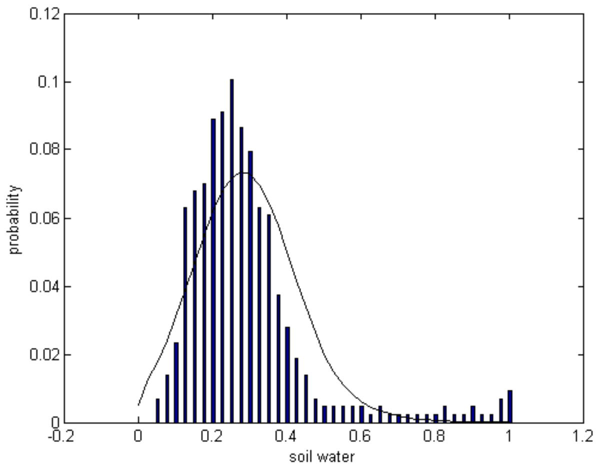

Fig. 3. The probability distribution function of soil water. The black bar represents the numerically generated data, and the line is for the analytically derived data.

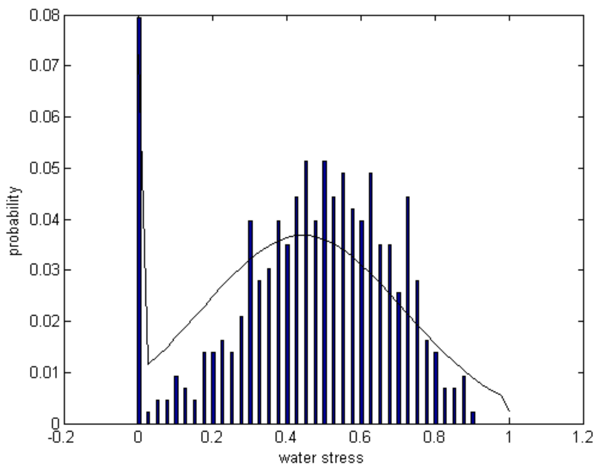

Fig. 4. The probability distribution function of water stress. The black bar represents the numerically generated data, and the line is for the analytically derived data.

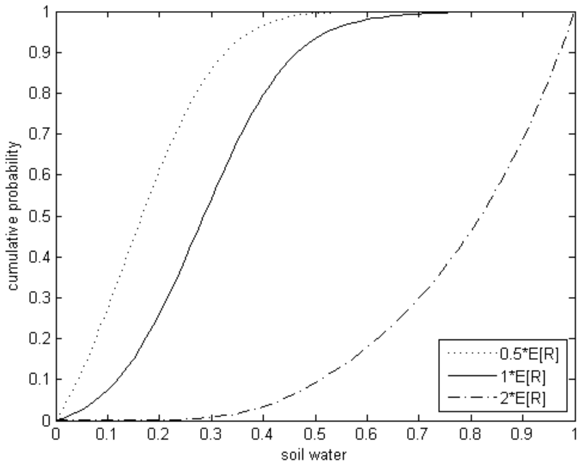

Fig. 5. The cumulative probability distribution function of soil water based on the changes in average daily precipitation rate.

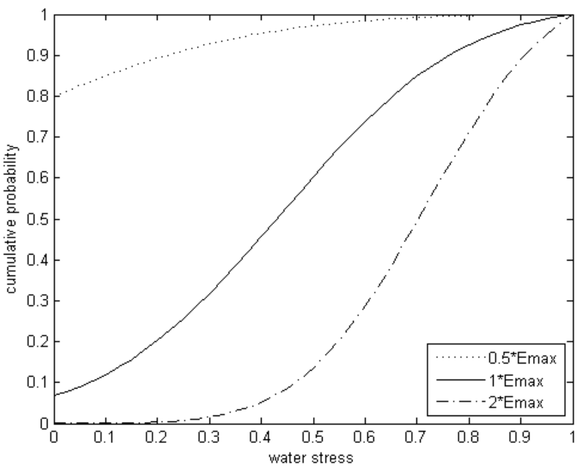

Fig. 6. The cumulative probability distribution function of plant water stress based on the changes in the potential evapotranspiration rate.

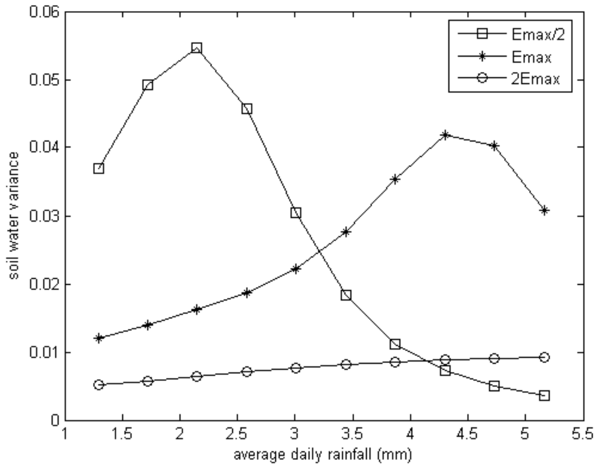

Fig. 7. Behaviors in the variance of soil water with different potential loss rates.

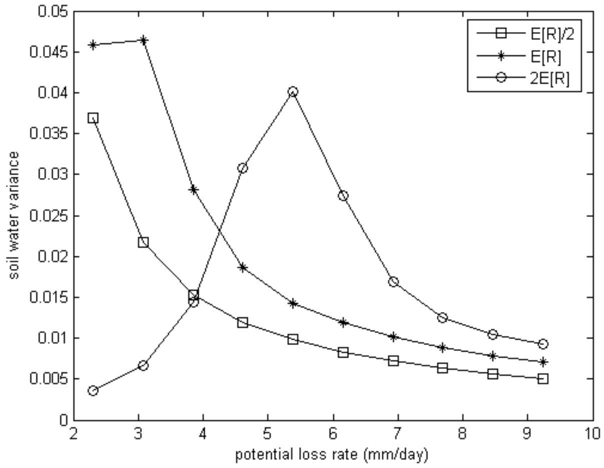

Fig. 8. Behaviors in the variance of soil water with different average daily rainfall rates.

would be changed quite differently. Fig. 6 describes the cumulative PDFs of plant water stress when the potential evapotranspiration rate was changed. Since the availability of soil water became less as the maximum evapotranspiration rate increases, the plant water stress has increased. Also, the possibility of vegetation with no stress soared dramatically when the maximum evapotranspiration rate reduces to half.

Fig. 7 illustrates how the variance of soil water behaved when the average rainfall rate was changed with different potential loss rates. As can be seen in Fig. 7, the variance of soil water was increased with the average daily rainfall rate until average daily rainfall rate reached the potential loss rate, and then, the variance is decreased even though the average daily rainfall rate increased. This can be explained by the relative dominance between rainfall and evapotranspiration given in the same soil and vegetation characteristics. In the rainfall-dominant case, the variance of soil water would be increased as average daily rainfall is increased. On the other hand, in the evapotranspiration dominant case, the variance of soil water would be decreased as average daily rainfall is increased. Such behavior is also represented in Fig. 8. The variance of soil water was increased with the potential loss rate until the potential loss rate reached average daily rainfall rate, and then, the variance was decreased even though the potential loss rate increased. In the rainfall-dominant case, the variance of soil water would be increased as the potential loss rate is increased. On the other hand, in the evapotranspiration dominant case, the variance of soil water would be decreased as the potential loss rate is increased.

4. Conclusions

In this study the PDF of soil water and plant water stress in steady-state condition using the cumulant expansion technique is analytically derived for the better understanding of their movements. The proposed model has the advantage of providing the probabilistic solution in the form of a PDF, from which one can find the ensemble average behavior of the system. The derived model is simple but can represent the important features of the system suitably and offer a cornerstone for further intensive research about the influences of soil water and plant water stress.

As a result, it is shown that the analytically derived steady-state soil water and plant water stress PDF in this study could make a good agreement to the numerically generated steady-state PDF from a soil water storage governing equation with rainfall forcing. Hence, the steady-state analysis is thought to be appropriate for the study of soil water and plant water stress dynamics where the seasonality of rainfall is not very significant.

The suggested model herein represented the overall characteristics in the rainfall-soil-vegetation system properly: (1) soil water consisted of the decreased steady-state PDF, and plant water stress has the increased steady-state PDF when rainfall is decreased; (2) when the evapotranspiration is increased, soil water is decreased and plant water stress is increased; and (3) the variability of soil water is affected by the relative dominance between rainfall and evapotranspiration given in the same soil and vegetation characteristics.

The major conclusion, however, is that the proposed simplified stochastic soil water and plant water stress model can provide quite a sensible explanation of the main soil water and plant water stress probabilistic properties even if only the rainfall variability is accounted for.

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