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Analytical solution of nonlinear Boussinesq equation

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ABSTRACT

The solution to the second-order unsteady partial differential one-dimensional Boussinesq equation is examined. The physical problem concerns unsteady flow in a semi-infinite unconfined aquifer bordering a lake. Two cases are examined: In the first case, a sudden rise and subsequent stabilization in the water level of the lake occurs, thus the aquifer is recharging from the lake. In the second-case, the lake sustains a surface drop, and the aquifer is discharging to the lake. In the first part of the article a new analytical Wiedeburg's transformed method for the solution of the one-dimensional Boussinesq equation is presented, for both recharging and discharging of a homogeneous unconfined aquifer. The solution is presented by a simple algebraic equation, transformed into a fourth-degree polynomial approximation for the head profiles. Additionally new formulas for recharged and discharged stored volumes are presented. Subsequently some applications are presented with profiles and stored volume, compared to other analytical solution. This extremely simple solution was proved to be very accurate in comparison to other analytical solutions in existence.

Keywords: Horizontal water flow; Aquifer; Charging; Discharging; Water volume; Boussinesq equation

1. Introduction

The horizontal water flow concerning unconfined aquifers without precipitation is described by the one dimensional second-order unsteady partial differential equation, called Boussinesq equation.

$$\frac{\partial h}{\partial t} = \frac{K}{S}\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) \tag{1}$$

This equation was presented by Boussinesq, with the following assumptions [1]: (a) the inertial forces are negligible and (b) the horizontal component of velocity u_x does not vary depending on depth, and it is a function of x and t. In 1904, Boussinesq provided a special solution of this nonlinear equation in the French Journal "Journal de Mathématiques Pures et Appliquées". Boussinesq's solution concerned the case of an aquifer overlying an impermeable layer, with boundary conditions like those of a soil drained by a drain installed in the impermeable substratum. A solution to Boussinesq's equation using the method of small disturbances was published by Polubarinova-Kochina [2–4]. An approximate closed form solution was obtained by Tolikas et al. [5], by applying similarity transformation and polynomial approximation. Lockington [6] provided a simple analytical solution using a weighted residual method. This method was applied to both, recharging and discharging of an unconfined aquifer, due to sudden change in the head at the origin. Moutsopoulos [7] derived

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an analytical approximate solution of the Boussinesq's equation by combining an expression describing the water table elevation upstream, obtained by the Adomian decomposition approach, to an existing polynomial expression [5], adequate for the downstream region. Lockington et al. analyzed the problem of flow in a one-dimensional semi-infinite horizontal aquifer, initially dry and a head expressed as power-law function of time at the origin [8]. A quadratic approximate solution was derived in agreement with numerical results. The traveling wave method was used by Basha [9], in order to obtain a nonlinear solution of a simple logarithmic form. The solution is adaptable to any flow situation that is recharge or discharge and allows results of practical importance in hydrology. Additionally algebraic equations are included for the velocity of the propagation front, wetting front position and relationship for aquifer parameters. Chor et al. [10] provided a series solution for the nonlinear Boussinesq equation in terms of the Boltzmann's transformation in a semi-infinite domain. A decomposition method was used by Jiang and Tang [11], separating the original problem into linear diffusion equations. They developed a general approximate explicit solution in terms of error functions. More recently Hayek provided an approximate solution, by introducing an empirical function with four parameters [12]. The parameters were obtained using a numerical fitting procedure performed with Microsoft Excel Solver. It should also be mentioned Moutsopoulos [23], who presented the flow processes in unconfined double porosity aquifers, and in the special case for single porosity aquifer, proposed a simple formula of the discharge flow rate. Several other authors including [13–19,20,22,23] provided useful tools for testing the accuracy of numerical methods.

In the present article, the problem of unsteady flow in a semi-infinite unconfined aquifer bordering a lake is examined. There is a sudden change and subsequent stabilization of the lake's water level, thus the aquifer is recharging from the lake or discharging to the lake. In the first part of the article a new analytical Wiedeburg's [21] transformed method to solve the one-dimensional Boussinesq equation is presented, for both the recharging and discharging of a homogeneous unconfined aquifer. The solution is expressed by a simple algebraic equation for the head profiles. Additionally algebraic equation is derived for the stored or drained water volume. Subsequently some applications are presented with profiles and stored volume, compared to other analytical solutions. The present solution was proved to be very accurate for small and intermediate values of the ratio value $h_{x=0}/h_{x\to\infty}$ concerning, the profiles propagation, as well the water stored volume, with respect to other exact analytical solutions.

2. Governing equations

The equation to be solved is:

$$\frac{\partial h}{\partial t}=\frac{K}{S}\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) \tag{2}$$

with initial and boundary conditions.

$$t=0, h(x,0)=h_0, x=0, h(0,t)=h_1, \quad x\to\infty, h(\infty,t)=h_0 \tag{3}$$

where h is the piezometric head, K is the hydraulic conductivity, S is the specific yield of the aquifer, h_0 is the initial piezometric head and h_1 is the piezometric head at the origin. A sudden increase (or decrease) of the piezometric head at the origin is considered ($h_1 > h_0$, or $h_1 < h_0$).

3. Analytical solution

3.1. Transformed Wiedeburg's solution

3.1.1. Profiles

Wiedeburg proposed a solution for a nonlinear heat diffusion problem in "Annalen der Physik" [21]. A transformed aspect of Wiedeburg's solution is given now for the case of Boussinesq equation. A new variable $H = h - h_1$ is introduced now in Eq. (2), and the above equation becomes (Appendix):

$$\frac{\partial H}{\partial t}=\frac{\partial}{\partial x}\left(\frac{Kh_1}{S}+\frac{KH}{S}\frac{\partial H}{\partial x}\right) \tag{4}$$

with new initial and boundary conditions.

$$\begin{aligned}&t=0 \quad H(x,0)=t_0=h_0-h_1,\\&x=0 \quad H(0,t)=0,\\&x=\infty \quad H(\infty,t)=t_0=h_0-h_1,\text{ or}\end{aligned} \tag{5}$$

$$\frac{\partial H}{\partial t}=\frac{\partial}{\partial x}\left[a_0(1+aH)\frac{\partial H}{\partial x}\right]=a_0\frac{\partial^2 H}{\partial x^2}+\frac{a_0 a}{2}\frac{\partial^2 H^2}{\partial x^2} \tag{6}$$

where the following relations have been inserted.

$$a_0=\frac{Kh_1}{S}, a=\frac{1}{h_1}, a_0 a=\frac{K}{S}, \mu=\frac{h_1}{h_0} \tag{7}$$

The linear form of Eq. (6) (for $a = 0$) is:

$$\frac{\partial H}{\partial t}=a_0\frac{\partial^2 H}{\partial x^2} \tag{8}$$

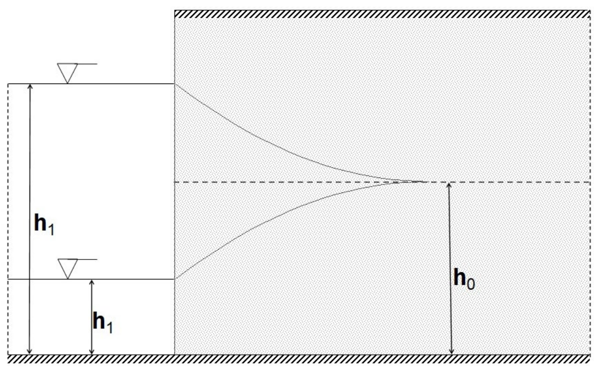

Fig. 1. Definition sketch of the investigated problem. This sketch is referred to Eq. (2). The level h_0 concerns the initial water level and the level h_1 concerns the sudden increase ($h_1 > h_0$), or decrease ($h_1 < h_0$) of the piezometric head at the origin.

with its solution.

$$H = t_0\left[1 - \text{erfc}\left(\frac{x}{2\sqrt{a_0 t}}\right)\right] \quad (9)$$

where a_0t and ξ are the Boltzmann's transformation:

$$a_0 t = \frac{Kh_1 t}{S}, \xi = \frac{x}{2\sqrt{a_0 t}} = \frac{x}{2\sqrt{Kh_1 t / S}} \quad (10)$$

Wiedeburg [21] suggested that for the term of Eq. (6) containing "a" a value of the h can be chosen, which would conform to the appropriate expression if "a" = 0. This substitution transforms the nonlinear equation into a linear one with a source dependent on the space coordinate and the time.

$$\frac{\partial H}{\partial t} = a_0 \frac{\partial^2 H}{\partial x^2} + \frac{a_0 a t_0^2}{2}\frac{\partial^2}{\partial x^2}\left[\text{erf}\frac{x}{2\sqrt{a_0 t}}\right]^2 \quad (11)$$

Now the Boltzmann's transformation $\xi = x / 2\sqrt{a_0 t}$ is introduced and an ordinary differential equation is obtained:

$$\frac{d^2 H}{d\xi^2} + 2\xi\frac{dH}{d\xi} = -\frac{a t_0^2}{2}\frac{d^2}{d\xi^2}\left[\text{erf}\xi\right]^2 \quad (12)$$

with new boundaries.

$$\xi = 0 \; H = 0, \xi = \infty \; H = t_0 \quad (13)$$

The solution of the equation is:

$$\begin{aligned} H = h - h_1 &= A + B\Phi - \left(a t_0^2 / \sqrt{\pi}\right)\left[1 - \Phi\right]\xi\exp\left(-\xi^2\right) \\ &+ \left(a t_0^2 / \pi\right)\left[1 - \exp\left(-2\xi^2\right)\right] - \frac{1}{2}\left(a t_0^2\right)\Phi^2, \\ \Phi &= \text{erfc}(\xi),\ A = t_0 - \left(a t_0^2 / \pi\right), B = a t_0^2\left(1/2 + 1/\pi\right) - t_0 \end{aligned} \quad (14)$$

In order to simplify Eq. (14), it is posed:

$$\begin{aligned} F(\xi) &= -\frac{1}{\pi} + \left(\frac{1}{2} + \frac{1}{\pi}\right)\Phi - \frac{1}{\sqrt{\pi}}(1 - \Phi)\xi\exp\left(-\xi^2\right) \\ &+ \frac{1}{\pi}\left(1 - \exp\left(-2\xi^2\right) - \frac{1}{2}\Phi^2\right. \end{aligned} \quad (15)$$

and the Eq. (14) becomes:

$$\frac{h - h_0}{h_0 - h_1} = \ell F(\xi) - \Phi(\xi) = \Omega(\mu, \xi), \ell = \frac{h_0 - h_1}{h_1} = \frac{1}{\mu} - 1, \text{ or} \quad (16)$$

$$h = h_0 + \left(h_0 - h_1\right)\Omega(\mu, \xi) \quad (17)$$

It is to be noted that Eqs. (14) and (17) both satisfy the boundary conditions (13) and (2). Indeed:

$$\begin{aligned} &\text{For } \xi = 0, F(\xi) = 0, \Phi(\xi) = 1, \frac{h - h_0}{h_0 - h_1} = -1 \rightarrow h = h_1, H = 0, \\ &\text{For } \xi = \infty, F(\xi) = 0, \Phi(\xi) = 0, \frac{h - h_0}{h_0 - h_1} = 0 \rightarrow h = h_0, \\ &\quad H = t_0 = h_0 - h_1 \end{aligned} \quad (18)$$

Function $F(\xi)$ is now expressed as a polynomial approximation vs. ξ, which is of the form:
$F(\xi) = \underset{0\le\xi\le0.349}{F_1(\xi)} \cup \underset{0.349<\xi\le\xi_w}{F_2(\xi)}$ (ξ_w is the water front and the value 0.340 is approximately the inflection point of $F(\xi)$):

$$F_i(\xi) = a_{i,0} + a_{i,1}\xi + a_{i,2}\xi^2 + a_{i,3}\xi^3 + a_{i,4}\xi^4 = \sum_{j=0}^{4} a_{i,j}\xi^j, \quad i = 1,2 \quad (19)$$

For Eq. (19) it is used a polynomial regression given by Excel, which provides the coefficients $a_{i,j}$, as well as the correlation coefficient R(R = 0.99999 for this case) and the polynomial equation. This approximation is presented in Fig. 2 and the absolute mean distance between the two functions $F(\xi)$ and $F_i(\xi)$ is about $O(10^{-6})$ for $F_1(\xi)$ and $O(10^{-4})$ for $F_2(\xi)$.

Each polynomial approximation (Eq. 19) has five coefficients which are presented in Table 1.

3.1.2. Stored volume/width

The stored volume/width is deduced from the following relation:

$$\begin{aligned} V &= \int_0^{x_w} S\left(h - h_0\right)dx = 2\sqrt{Kh_1 St}\left(h_0 - h_1\right) \\ &\int_0^{\xi_0} \Omega(\mu, \xi)d\xi \;\left(x_w = \text{waterfront}\right) \end{aligned} \quad (20)$$

It is posed now.

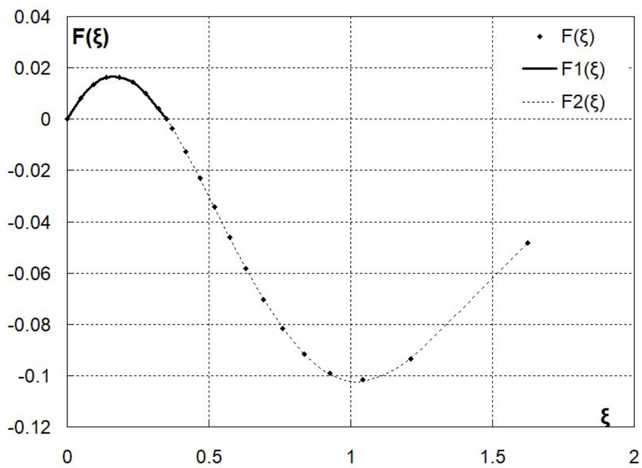

Fig. 2. Functions $F(\xi)$, $F_i(\xi)$ with respect to ξ.

Table 1
Coefficients $a_{i,j}$, of polynomial approximation, ($i = 0, \ldots, 4$, $j = 1, 2$)

	$a_{0,j}$	$a_{1,j}$	$a_{2,j}$	$a_{3,j}$	$a_{4,j}$
$F_{i=1}(\xi)$	–0.00000422	0.205319663	–0.643143783	–0.018209237	0.504985372
$F_{i=2}(\xi)$	–0.00110088	0.284398024	–1.114903246	0.988794972	–0.259307367

Table 2
Parameters a, c, m, n

μ	a	c	m	n
1.5	–5.956736	–0.251981	1.113585	1.373567
3	–3.198256	–0.192291	1.136821	1.417222
10	–1.008772	–0.406052	1.248848	1.111019

3.1.2.1. Recharging case

$$H(\mu) = \int_0^{\xi_w} \Omega(\mu,\xi)\,d\xi = \left(\frac{1}{\mu}-1\right)\int_0^{\xi_w} F(\xi)\,d\xi - \int_0^{\xi_w} \Phi\,d\xi$$
$$= -0.000832602\mu^4 + 0.012299387\mu^3 - 0.0694412416\mu^2 + 0.18784065\mu - 0.692301119, \quad \mu = h_1/h_0 \quad (21a)$$

3.1.2.2. Discharging case

$$H(\mu') = -0.09647(\mu') - 0.46562,\ \mu' = 1/\mu \quad (21b)$$

Numerical results for the function $H(\mu)$ with respect to μ are presented in Figs. 3 and 4. The substitution of Eq. (21a) into Eq. (20) gives:

$$V = \int_0^{x_w} S(h-h_0)\,dx = 2H(\mu)(h_0-h_1)\sqrt{Kh_1St}$$
$$= -2H(\mu)\sqrt{\mu}(\mu-1)\sqrt{Kh_0^3St} = C_r^w\sqrt{Kh_0^3St}, \quad (22)$$

where:

$$C_r^w = -2H(\mu)\sqrt{\mu}(\mu-1) = \text{Dimensionless recharge coefficient} \quad (23)$$

and Eq. (22) takes its final form.

$$V = C_r^w\sqrt{Kh_0^3St}\left(\text{m}^3/\text{m}\right) \quad (24)$$

3.1.3. Flow-rate Q

$$Q = \frac{dV}{dt} = \frac{1}{2}C_r^w\sqrt{\frac{Kh_0^3S}{t}}\left(\text{m}^3/\text{m/d}\right) \quad (25)$$

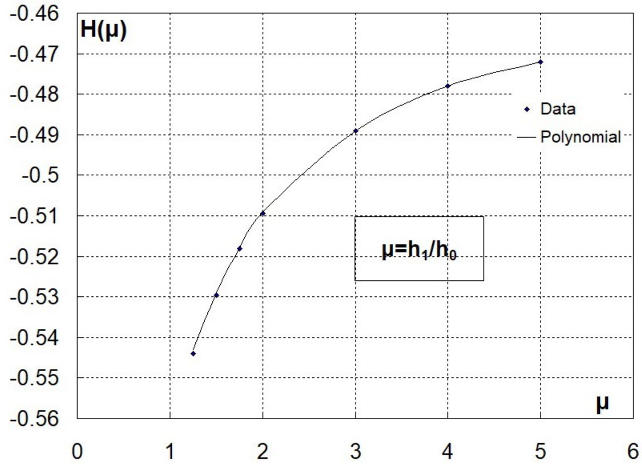

Fig. 3. Function $H(\mu)$ with respect to μ (recharging case).

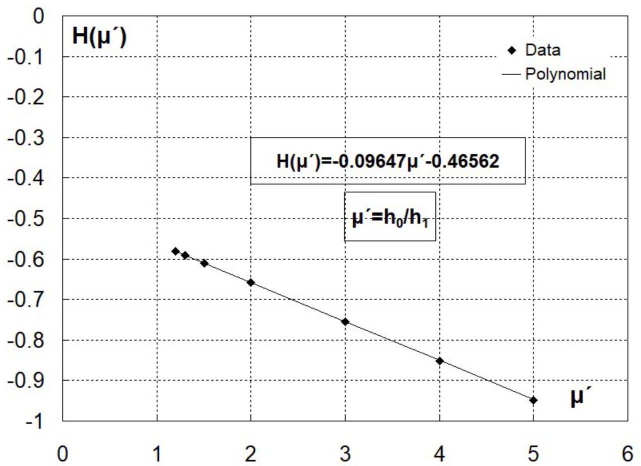

Fig. 4. Function $H(\mu')$ with respect to $\mu' = h_0/h_1$ (discharging case).

3.2. Applications

3.2.1. First example

3.2.1.1. Recharging

Profiles

To make a benchmark test for the recharging aquifer proposed solution and other existing analytical solutions, four other methods are chosen and a numerical method, which is assumed to be the reference solution. The numerical solution presented in Moutsopoulos [7] and Hayek [12] is the Runge–Kutta method. The four chosen methods are: Polubarinova-Kochina method [4], which was first reported by Moutsopoulos [7]. Lockington method [6], Moutsopoulos method [7] and Hayek method [12]. In order to test also the nonlinear character of the equation, the solution has been evaluated for three different values of the ratio $\mu = h_1/h_0 = 1.5, 3, 10$. The same example with [6], [7] and [12] is used, namely the estimation of a water table for an aquifer with $K = 20.00$ m/d, $S = 0.27$ and $t = 5$ d, and ($h_1 = 3$ m, $h_0 = 2$ m), ($h_1 = 3$ m, $h_0 = 1$ m), ($h_1 = 10$ m, $h_0 = 1$ m).

Polubarinova-Kochina method [4]

Polubarinova-Kochina's expansion of third-order is examined, [7] vs. the present solution, which was first reported by Moutsopoulos [7]. This expansion corresponds to Eq. (18) of [7] plus an additional term $\ell^3 u_3$.

Lockington method [6]

An approximate analytical solution of Lockington [6], who has solved the nonlinear Boussinesq equation by a weighted residual approach and his solution can be also applied for recharging and discharging aquifers. Lockington's solution is:

$$h = h_0 + \left(h_1 - h_0\right)\left(1 - \frac{x}{\lambda}\sqrt{\frac{S}{Kt}}\right)^{1/\mu_1} \quad (26)$$

The parameters λ, μ_1, for the case $\mu = 1.5$ are given by Lockington ($\lambda = 5.3453$, $\mu_1 = 0.5215$). For the other values of $\mu = 3$ and $\mu = 10$, the numerical data was taken of [7], because the relations given the parameters λ, μ_1 in [6] are complicated.

Moutsopoulos method [7]

Moutsopoulos [7] derived an analytical approximate solution of the Boussinesq equation by combining an expression describing the water table elevation upstream, obtained by the Adomian's decomposition approach, to an existing polynomial expression [5]. The numerical data for $\mu = 1.5, 3, 10$ were taken from his article.

Hayek method [12]

Hayek solution for recharging case is:

$$x_r = a\sqrt{\left(Kh_0 / St\right)} \left\{ \begin{aligned} &\left[-m + nc\left(h/h_0 - 1/\mu\right)^n\right]\left(h/h_0 - 1\right)^{m-1} e^{-c\left(1-h/h_0\right)^n} \\ &+\left[m - nc\left(1 - 1/\mu\right)^n\right]\left(1 - 1/\mu\right)^{m-1} e^{-c\left(1-\mu\right)^n} \end{aligned} \right\} \quad (27)$$

where h/h_0, for recharging $\mu = h_0/h_\infty$, ($h_0 = h_1$, $h_\infty = h_0$).

In order to find the water table profiles, the function $X_r = F(h, a, c, m, n)$ was solved for the three different values of $\mu = 1.5, 3, 10$ and the parameters a, c, m, n were taken from Hayek [12]:

It is noteworthy that the parameter "a" for $\mu = 10$, should take the value $a = -0.829800$ for the best results.

Fig. 5a–c illustrate the profiles of the solutions for the three values of μ:

The stored volume/width recharge coefficient C_r:

For the numerical value of the dimensionless recharge coefficient C_r Eq. (24) is applied:

$$V = C_r^i \sqrt{Kh_0^3 t / S}\left(\mathrm{m}^3/\mathrm{m}\right) \quad (28)$$

where i indicates the different solutions formulae. For the case $\mu = h_1/h_0 = 1.5$ and applying Fig. 3, $C_r^w = 0.648$ is found, which is in a good agreement with the numerical value of Table 3 for the case of this study.

3.2.1.2. Discharging

Profiles

For the discharging case the same example with [6] is used, namely the estimation of a water table for an aquifer, with $K = 20.00$ m/d, $h_1 = 3$ m, $h_0 = 2$ m and $t = 5$ d, $S = 0.27$, ($h_0 = 3$, $h_1 = 2$), so:

Water profiles in the present study are:

$$h = h_0 + \left(h_0 - h_1\right)\Omega\left(\mu, \xi\right) = 3 + \Omega\left(\mu, \xi\right) \quad (29)$$

Water profiles for the Lockington's case are:

$$h = 3 - \left(1 - 0.006028x\right)^{3.0328} \quad (30)$$

Except to Lockington's case, it was difficult to find available data in order to make a complete benchmark test. Consequently, for the case $\mu = 10$, our research was limited to make a comparison between Runge–Kutta and Wiedeburg's transformation method, but failed to give sufficient results.

Fig. 6 illustrates the water profiles of the three methods and Table 4 shows the performance of analytical solutions for $t = 5$ d, and for the ratio $\mu = h_1/h_0 = 1.5$.

The results are provided by numerical integration of the water profiles. The application of Eqs. (21b) and (24) gives for this study:

$$V = C_d^w \sqrt{Kh_0^3 St}\left(\mathrm{m}^3/\mathrm{m}\right), C_d^w = 2H\left(\mu\right)\left(1 - 1/\mu\right)\mu\sqrt{\mu}, \mu = 2/3 \quad (31)$$

in which $H\left(h_0/h_1\right)\big|_{\text{discharge}} = H\left(1.5\right) = -0.61628$, $C_d^w = 0.33646$ and $V = 9.099$ m^3/m.

For Lockington's solution it is:

$$V = C_d^L \sqrt{Kh_0^3 St}, \left(\mathrm{m}^3/\mathrm{m}\right), C_d^L = \frac{\lambda\mu_1}{1-\mu_1}\frac{1-\mu}{\sqrt{h_0}}, \sqrt{Kh_0^3 tS} = 27 \quad (32)$$

where $\lambda = -8.5535$, $\mu_1 = -0.2543$, $C_d^L = 0.3337$, $V = 9.0099$ m^3/m.

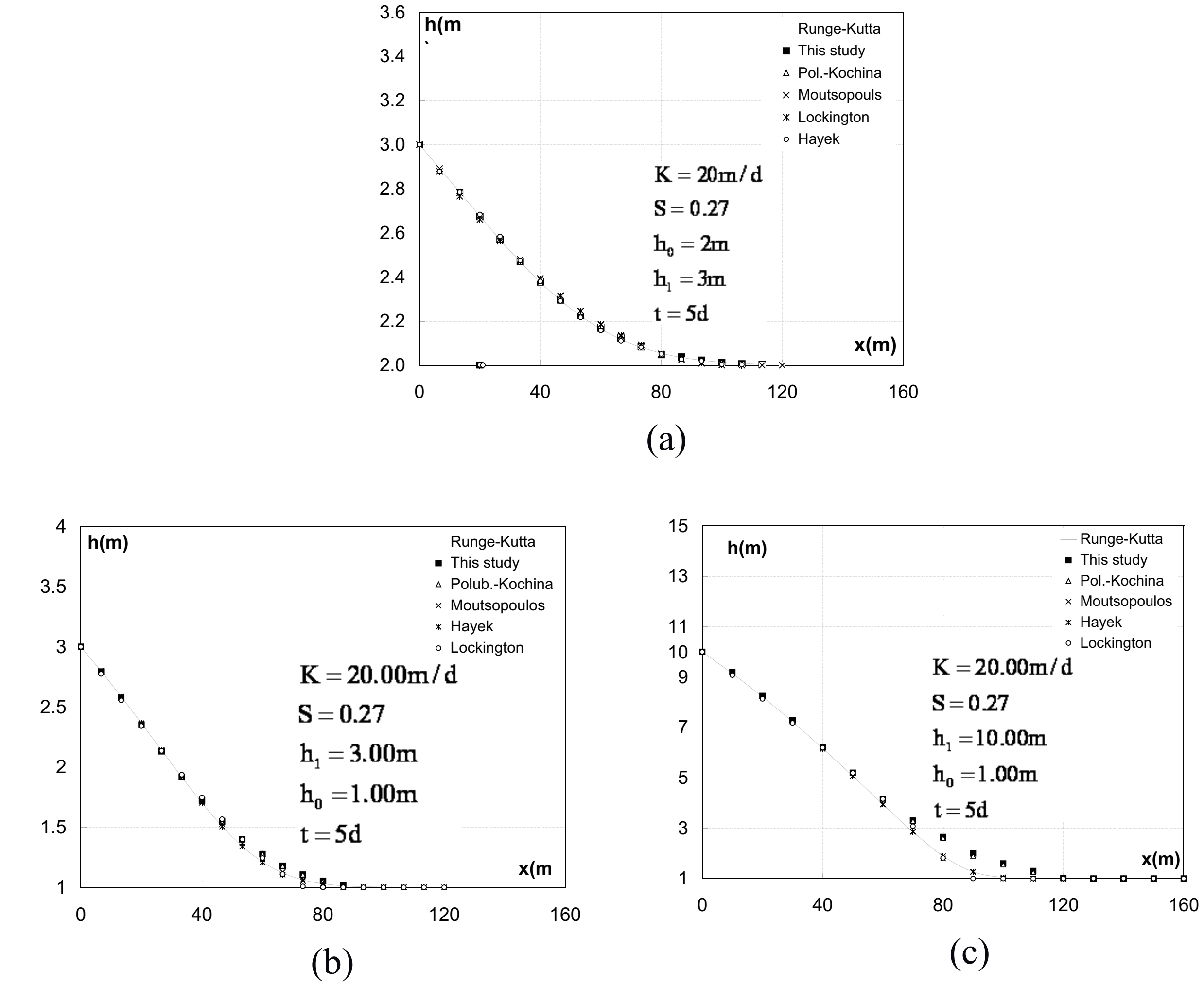

Fig. 5. (a) μ = 1, (b) μ = 3, and (c) μ = 10. Water table profiles h vs. x at t = 5 d and for three different values of the ratio $\mu = h_1/h_0$. The illustrated profiles are: Runge–Kutta (solid line), this study (black squares), Polubarinova-Kochina (triangles), Moutsopoulos (×), Hayek (Ж), Lockington (°).

This is a very good agreement between numerical and analytical results.

3.2.2. Second example

3.2.2.1. Recharging

In order to test the present solution for the case of larger values, a shallow sand-gravel aquifer is considered with the following hydraulic characteristics: hydraulic conductivity K = 300 m/d, specific yield S = 0.15, h_0 = 30 m, h_1 = 45 m, t = 100 d. The aquifer is semi-infinite and underlained by a horizontal impermeable datum, having an initial water table elevation h_0 = 30 m. In this example the case of a sudden elevation of the water level to h_1 = 45 m at the origin is examined. To test the validity of present solution, our research was limited to make a comparison between Runge–Kutta method (reference solution), this study method and Polubarinova-Kochina method, due to lack of available data. The water table profile of this study can be written as follows:

(Eq. 18), $h(\mu,t) = h(1.5,100) = 30–15\Omega(1.5,\xi)$

Fig. 7 illustrates the profiles of water table for t = 100 d of the present solution vs. Runge–Kutta method and Polubarinova-Kochina's expansion ([7]). In Table 5 it is shown that the three solutions approach each other closely, and the absolute relative mean error D_{Cr} is of the order $O(10^{-3})$.

The stored volume/width:

The results are provided by numerical integration of the water profiles. The application of Eqs. (21a) and (24) gives for this study:

$$V = C_d^w \sqrt{K h_0^3 S t}\left(\mathrm{m^3/m}\right), C_d^w = 2H(\mu)\left(1-1/\mu\right)\mu\sqrt{\mu}, \mu = 2/3 \tag{33}$$

where C_2^w = 0.647789, V = 7,140.38 m^3/m.

It is a very good agreement between theoretical and numerical methods.

Table 3
Performance of various analytical solutions for t = 5 d, and for three different values of the ratio $\mu = h_1/h_0$ = 1.5, 3, 10

		R-K vs. This study	R-K vs. Pol.-Kochina		R-K vs. Moutsopoulos		R-K vs. Lockington		R-K vs. Hayek
	D_{Cr}	2.31E-03	5.21E-04		1.07E-02		1.33E-03		2.11E-02
	D	2.27E-06	7.64E-07		2.84E-05		3.37E-05		8.58E-06
			Runge–Kutta	This study	Pol.-Kochina	Moutsopoulos	Lockington	Hayek	
μ = 1.5	V		9.516	9.548	9.511	9.618	9.529	9.717	
	C_r		0.648	0.649	0.647	0.654	0.648	0.661	
	Q		0.952	0.955	0.951	0.962	0.953	0.972	
		R-K vs. This study	R-K vs. Pol.-Kochina		R-K vs. Moutsopoulos		R-K vs. Lockington		R-K vs. Hayek
	D_{Cr}	1.39E-03	2.02E-02		2.43E-03		2.15E-02		1.13E-04
	D	6.51E-04	6.34E-04		9.13E-05		1.24E-06		7.61E-04
			Runge–Kutta	This study	Pol.-Kochina	Moutsopoulos	Lockington	Hayek	
μ = 3	V		17.646	18.009	18.002	17.689	18.026	17.644	
	C_r		3.396	3.391	3.465	3.404	3.469	3.396	
	Q		1.765	1.801	1.800	1.769	1.803	1.764	
		R-K vs. This study	R-K vs. Pol.-Kochina		R-K vs. Moutsopoulos		R-K vs. Lockington		R-K vs. Hayek
	D_{Cr}	8.08E-02	7.18E-02		3.99E-04		1.83E-03		4.84E-04
μ = 10	D	5.23E-02	4.18E-02		2.50E-04		3.39E-03		9.44E-06
			Runge–Kutta	This study	Pol.-Kochina	Moutsopoulos	Lockington	Hayek	
	V		111.594	120.609	119.610	111.639	111.799	111.540	
	C_r		21.476	23.211	23.019	21.485	21.516	21.466	
	Q		11.159	12.061	11.961	11.164	11.180	11.154	

C_r = dimensionless recharge coefficient (23), V = stored volume/width, $D_{Cr} = \mathrm{abs}\left(C_r^{\text{Runge-Kutta}} - C_r^i\right) / C_r^{\text{Runge-Kutta}}$, C_r^i = remainder dimensionless recharge coefficients, V = stored volume/width, $D = \frac{1}{n}\sum_{i=0}^{n}\left(\left|h^{\text{Runge-Kutta}} - h^i\right| / h^{\text{Runge-Kutta}}\right)^2$, h^i = remainder water tables and Q = flow rate.

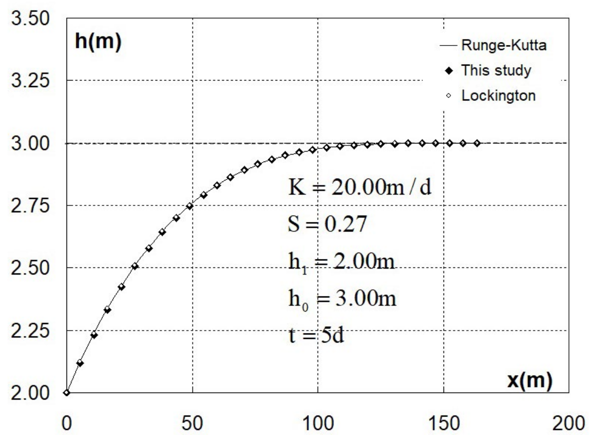

Fig. 6. Profiles of water table for t = 5 d.

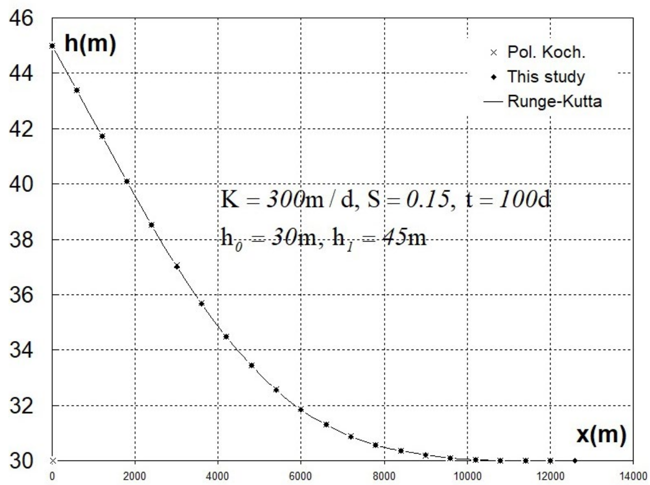

Fig. 7. Profiles of water table for t = 100 d (This study vs. Polubarinova-Kochina).

4. Discussion and conclusions

Eq. (18) has been written in the following simple expression, where:

$$\Omega(\mu,\xi) = (1/\mu - 1)F_i(\xi) - \Phi(\xi), \tag{34}$$

$$\frac{h - h_0}{h_0 - h_1} = \ell F(\xi) - \Phi(\xi) = \Omega(\mu,\xi), \ell = \frac{h_0 - h_1}{h_1} = \frac{1}{\mu} - 1 \tag{35}$$

Table 4
Performance of analytical solutions for t = 5d, and for the ratio $\mu = h_1/h_0 = 1.5$

	This study	Runge–Kutta	Lockington
V	9.099	9.063	9.010
C_r	0.337	0.336	0.334
Q	0.909	0.906	0.901
	R-K vs. T_s		R-K vs. L
	3.96E-03		5.86E-03

Table 5
Performance of analytical solutions for t = 5 d, and for the ratio $\mu = h_1/h_0 = 1.5$

	This study	Runge–Kutta	Lockington
V	7,141.878	7,137.288	7,157.720
C_r	0.648	0.648	0.649
Q	35.709	35.686	35.789
	R-K vs. T_s		R-K vs. L
	2.21E-03		2.85E-03

This function is very easy to calculate and gives immediately a value $h(\xi, \mu)$ for every value of μ and ξ. So, with the Boltzmann's transformation, the profiles of h vs. x for t constant are calculated according to the above simple transformed formula.

4.1. For the first example

4.1.1. Recharging case

According to Table 3 it is shown that for the cases:

- μ = 1.5 all methods give good results regarding the water tables propagation, the stored volume/width V and the non-dimensional recharge coefficient C_r, with a slightly smaller relative error of Polubarinova-Kochina, this study and Lockington methods. Besides the numerical values and the theoretical values of recharge coefficient C_r, show very good agreement for the cases of this study and Lockington.
- μ = 3 all methods gives good results regarding the water tables propagation, the stored volume/width V and the non-dimensional recharge coefficient C_r, with a slightly smaller relative error of Moutsopoulos and Hayek methods.
- μ = 10 the Moutsopoulos and Hayek methods show a strong smaller relative error vs. the methods of Polubarinova-Kochina, and this study and a slightly smaller relative error vs. Lockington method.

4.1.2. Discharging case

According to Table 4 the three methods, this study, Runge–Kutta and Lockington give good results regarding the water tables propagation, the stored volume/width V, and the non-dimensionnel recharge coefficient C_r, for the case μ = 1.5. The D_{Cr} relative errors of the Lockington's and this study methods vs. Runge–Kutta method are of $O(10^{-3})$. For large values of μ = 10 this study gives non-physical results.

4.2. For the 2nd example

4.2.1. Recharging case

A case for values of $h_0 = 30$, $h_1 = 45$ (μ = 1.5), is examined in order to test the present solution for large values. According to Table 5, the three examined methods Polubarinova-Kochina, this study and Runge–Kutta give good agreement regarding the water profiles (Fig. 7), stored volumes/width and non-dimensional recharge coefficients C_r. The D_{Cr} relative errors of the Polubarinova-Kochina and this study methods vs. Runge–Kutta method are of $O(10^{-3})$.

The present analytical solution is valid for small and intermediate values of the ratio μ. For these cases this study provides a simple and accurate formula allowing managers and engineers to solve practical problems (in irrigation and water management), and estimate the stored water of the aquifers, knowing only the recharge coefficient, the time and the initial parameters of the aquifer. For larger values the [7] and [12] can be used.

Finally the solution of this problem (sudden water level rise), are useful for the computation of flow volumes stored or extracted from the aquifers, recharging man-made by irrigation trenches (artificial recharge). According to Hayek [12] and Revelli and Ridolfi [22], water level changes in trenches/channels occur in times that are much shorter than those occurring in phreatic aquifers and therefore can be schematized by sharp jumps. Consequently approximate analytical solutions for problems with sudden water level rise are important in modeling such real cases.

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Appendix

Transformed Wiedeburg's solution [21]

A transformed aspect of Wiedeburg's [21] solution is given now for the case of Boussinesq equation. A new variable $H = h - h_1$ is introduced now in Eq. (1), and the equation becomes:

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left(\frac{Kh_1}{S} + \frac{KH}{S}\frac{\partial H}{\partial x}\right) \tag{A1}$$

with new initial and boundary conditions.

$$\begin{aligned} &t = 0\ H(x,0) = t_0 = h_0 - h_1, \\ &x = 0\ H(0,t) = 0,\ \ x = \infty\ H(\infty,t) = t_0 = h_0 - h_1, \text{ or} \end{aligned} \tag{A2}$$

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x}\left[a_0(1 + aH)\frac{\partial H}{\partial x}\right] = a_0\frac{\partial^2 H}{\partial x^2} + \frac{a_0 a}{2}\frac{\partial^2 H^2}{\partial x^2} \tag{A3}$$

where the following relations have been inserted:

$$a_0 = \frac{Kh_1}{S}, a = \frac{1}{h_1}, a_0 a = \frac{K}{S}, \mu = \frac{h_1}{h_0} \tag{A4}$$

The linear form of Eq. (3) (for $a = 0$) is:

$$\frac{\partial H}{\partial t} = a_0\frac{\partial^2 H}{\partial x^2} \tag{A5}$$

with its solution.

$$H = t_0\left[1 - \mathrm{erfc}\left(\frac{x}{2\sqrt{a_0 t}}\right)\right] \tag{A6}$$

where $a_0 t$ and ξ are the Boltzmann's transformation.

$$a_0 t = Kh_1 t / S, \xi = \frac{x}{2\sqrt{a_0 t}} = \frac{x}{2\sqrt{Kh_1 t / S}} \tag{A7}$$

Wiedeburg [21] suggested that it can be chosen for the term of Eq. (A3) containing 'a', a value of the H which would conform to the appropriate expression if 'a' = 0. This substitution transforms the nonlinear equation into a linear one with a source dependent on the space coordinate and the time.

$$\frac{\partial H}{\partial t} = a_0\frac{\partial^2 H}{\partial x^2} + \frac{a_0 a t_0^2}{2}\frac{\partial^2}{\partial x^2}\left[\mathrm{erf}\frac{x}{2\sqrt{a_0 t}}\right]^2 \tag{A8}$$

It is introduced now the Boltzmann's transformation $\xi = \frac{x}{2\sqrt{a_0 t}}$ and an ordinary differential equation is obtained:

$$\frac{d^2 H}{d\xi^2} + 2\xi\frac{dH}{d\xi} = -\frac{at_0^2}{2}\frac{d^2}{d\xi^2}\left[\mathrm{erf}\xi\right]^2 \tag{A9}$$

with new boundaries.

$$\xi = 0\ H = 0, \xi = \infty\ H = t_0 \tag{A10}$$

According to Wiedeburg [21] the solution of the equation is of the form:

$$H = \varphi_1 + \varphi_2\Phi(\xi), \Phi(\xi) = \mathrm{erfc}(\xi) \tag{A11}$$

After substitution in Eq. (A9), it becomes apparent that the functions φ_1, φ_2 are determined by the equations:

$$\frac{d\varphi_1}{d\xi} = -\Phi\frac{d\varphi_2}{d\xi} \text{ and } \frac{d\varphi_2}{d\xi}\frac{d\Phi}{d\xi} = -\frac{at_0^2}{2}\frac{d^2(1-\Phi)^2}{d\xi^2} \tag{A12}$$

The solution of Eq. (A12) yields:

$$\begin{aligned} \varphi_1(\xi) = A + at_0^2\xi^2\Phi + at_0^2\frac{1-\Phi}{2} - \left(at_0^2/\sqrt{\pi}\right)\xi e^{-\xi^2} \\ - at_0^2\xi^2\Phi^2 - at_0^2\frac{1-\Phi^2}{2} + 2at_0^2\left(\Phi/\sqrt{\pi}\right)e^{-2\xi^2} \end{aligned} \tag{A13}$$

$$\varphi_2(\xi) = B - at_0^2\xi^2 + at_0^2\xi^2\Phi + at_0^2\frac{1-\Phi}{2} - \left(at_0^2/\sqrt{\pi}\right)\xi e^{\xi^2} - at_0^2\Phi^2 \tag{A14}$$

Hence:

$$H = h - h_1 = A + B\Phi - \left(at_0^2/\sqrt{\pi}\right)\left[1-\Phi\right]\xi e^{-\xi^2} + \left(at_0^2/\pi\right)\left[1-e^{-2\xi^2}\right] - \frac{1}{2}\left(at_0^2\right)\Phi^2 \tag{A15}$$

Combining now the solution Eq. (A15) with boundary conditions (10) yields:

$$A = t_0 - \left(at_0^2/\pi\right), B = at_0^2\left(1/2+1/\pi\right) - t_0, \Phi = \mathrm{erfc}(\xi) \tag{A16}$$

and finally.

$$\frac{h-h_0}{h_0-h_1} = \ell F(\xi) - \Phi(\xi), \ell = \frac{h_0-h_1}{h_1} = \frac{1}{\mu} - 1 \tag{A17}$$